钱学森 力学手稿

1

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出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰100周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱(初编)》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

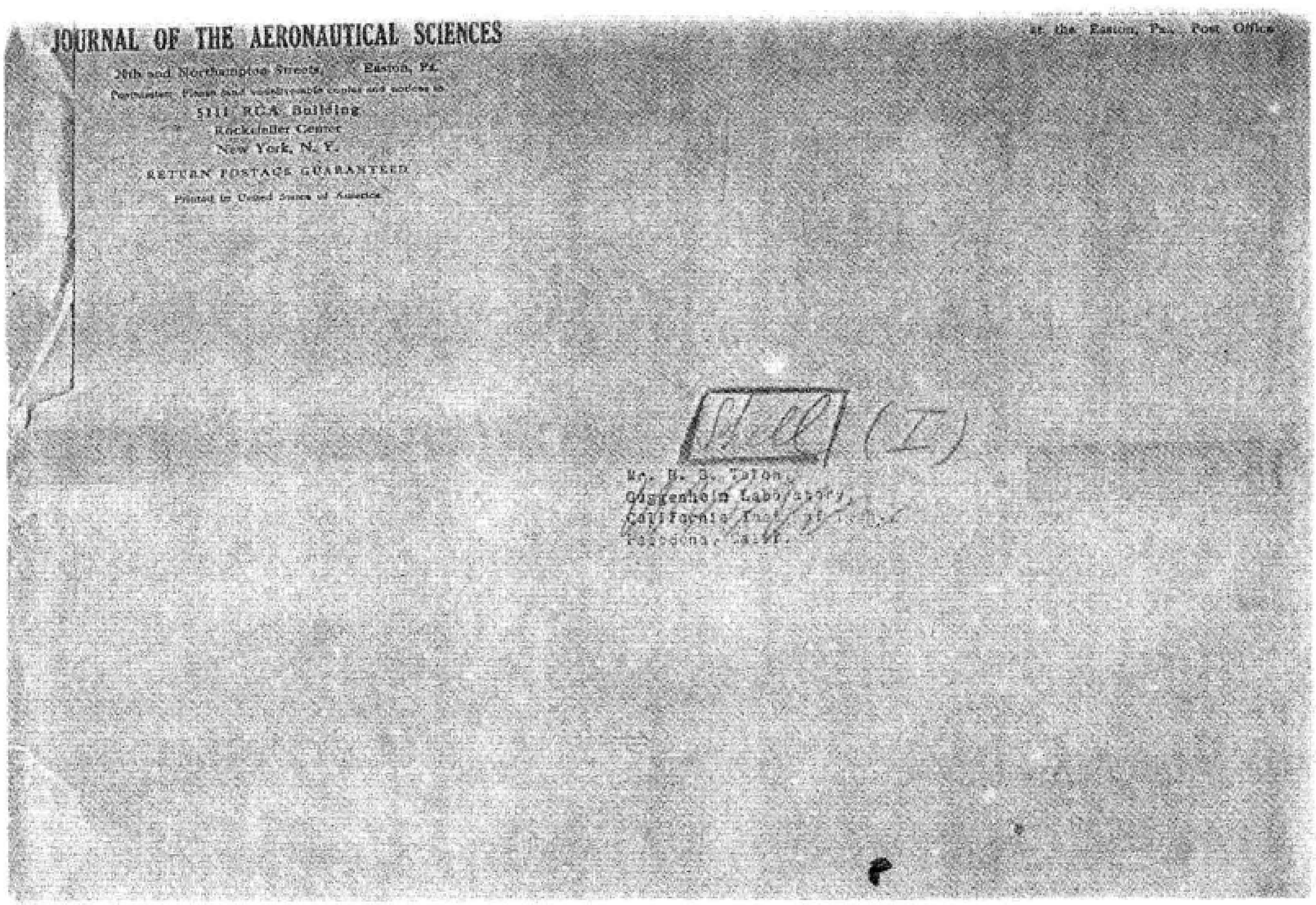
《钱学森力学手稿》共八册，其中《钱学森力学手稿1》包含四部分内容：Shell (I) Buckling of Cylindrical Shell without Shear; Shell (II) Collapse of Slightly Curved Circular Plate; Shell (III) Preliminary Calculation of Circular Cylinder; Buckling of Spherical Shell。其余七册将在之后陆续出版。

本手稿是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Contents

Section 1

Shell (Ⅰ) Buckling of Cylindrical Shell without Shear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Shell (I)

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California Inst. of Tech.,
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Buckling of Cylindrical Shell Without Shear

$$\frac{\partial N_x}{\partial x} = 0$$

$$\frac{\partial N_y}{\partial \theta} + a N_x \frac{\partial^2 v}{\partial x^2} + \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{a\,\partial \theta} = 0$$

$$a N_x \frac{\partial^2 w}{\partial x^2} + N_y + a\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{yx}}{\partial x \partial \theta} + \frac{\partial^2 M_y}{a\,\partial \theta^2} = 0.$$

$$\frac{\partial^2 u}{\partial x^2} + \nu\left\{\frac{\partial^2 v}{a\,\partial x \partial \theta} - \frac{1}{a}\frac{\partial w}{\partial x}\right\} = 0.$$

$$\frac{Eh}{1-\nu^2}\left\{\frac{\partial^2 v}{a\,\partial \theta^2} - \frac{1}{a}\frac{\partial w}{\partial \theta} + \nu\frac{\partial^2 u}{\partial x \partial \theta}\right\} - a N_x \frac{\partial^2 v}{\partial x^2}$$

$$+ D(1-\nu)\frac{1}{a}\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^3 w}{\partial x^2 \partial \theta}\right\} + \frac{D}{a}\left\{\frac{1}{a^2}\left(\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3}\right) + \nu\frac{\partial^3 w}{\partial x^2 \partial \theta}\right\} = 0.$$

$$\frac{\partial^2 v}{a\,\partial \theta^2} - \frac{1}{a}\frac{\partial w}{\partial \theta} + \nu\frac{\partial^2 u}{\partial x \partial \theta} + \alpha\left\{(1-\nu)a\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^3 w}{\partial x^2 \partial \theta}\right)\right.$$

$$\left. + \frac{1}{a}\left(\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^3 w}{\partial \theta^3}\right) + \nu a\frac{\partial^3 w}{\partial x^2 \partial \theta}\right\} - a\phi\frac{\partial^2 v}{\partial x^2} = 0.$$

or

$$\frac{\partial^2 v}{a\,\partial \theta^2} - \frac{1}{a}\frac{\partial w}{\partial \theta} + \nu\frac{\partial^2 u}{\partial x \partial \theta} + \alpha\left\{a(1-\nu)\frac{\partial^2 v}{\partial x^2} + a\frac{\partial^3 w}{\partial x^2 \partial \theta} + \frac{\partial^2 v}{a\,\partial \theta^2} + \frac{\partial^3 w}{a\,\partial \theta^3}\right\} - a\phi\frac{\partial^2 v}{\partial x^2} = 0$$

2)

$$\frac{\partial^2 u}{\partial x^2} + \nu\left\{\frac{\partial^2 v}{a\partial x\partial\theta} - \frac{1}{a}\frac{\partial w}{\partial x}\right\} = 0.$$

$$\frac{\partial^2 v}{a\partial\theta^2} - \frac{\partial w}{a\partial\theta} + \nu\frac{\partial^2 u}{\partial x\partial\theta} + \alpha\left\{a(1-\nu)\frac{\partial^2 v}{\partial x^2} + a\frac{\partial^3 w}{\partial x^2\partial\theta} + \frac{\partial^2 v}{a\partial\theta^2} + \frac{\partial^3 w}{a\partial\theta^3}\right\} - a\phi\frac{\partial^2 v}{\partial x^2} = 0.$$

$$-a\phi\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial u}{\partial x} + \frac{\partial v}{a\partial\theta} - \frac{w}{a} - \alpha\left[\frac{\partial^3 v}{a\partial\theta^3} + (2-\nu)a\frac{\partial^3 v}{\partial x^2\partial\theta} + a^3\frac{\partial^4 w}{\partial x^4}\right.$$

$$\left. + \frac{\partial^4 w}{a\partial\theta^4} + 2a\frac{\partial^4 w}{\partial x^2\partial\theta^2}\right] = 0.$$

$$-A\left(\frac{m\pi}{l}\right)^2 + \nu\left\{-B\frac{n}{a}\left(\frac{m\pi}{l}\right) - \frac{C}{a}\frac{m\pi}{l}\right\} = 0.$$

$$A\lambda^2 + B\cdot\nu n\lambda + C\nu\lambda = 0$$

or $\cancel{\lambda A + \nu n\lambda B}$

$$\underline{\lambda A + \nu n B + \nu C = 0}$$

$$\cancel{-\frac{B}{an^2}\quad C}\quad -\frac{n^2 B}{a} - \frac{nC}{a} - \nu A\cdot n\left(\frac{m\pi}{l}\right)$$

$$+\alpha\left\{a(1-\nu)(-1)B\cdot\left(\frac{m\pi}{l}\right)^2 - anC\left(\frac{m\pi}{l}\right)^2 - \frac{1}{a}n^2B - \frac{n^3C}{a}\right\} + Ba\phi\left(\frac{m\pi}{l}\right)^2 = 0.$$

$$n^2B + nC + \nu An\lambda + \alpha\left\{(1-\nu)B\lambda^2 + nC\lambda^2 + n^2B + n^3C\right\}$$

$$-B\phi\lambda^2 = 0.$$

$$\nu n\lambda A + \left(n^2 + \alpha n^2 + \alpha(1-\nu)\lambda^2 - \phi\lambda^2\right)B + \left(n + \alpha n\lambda^2 + \alpha n^3\right)C = 0.$$

3)

$$\lambda A + \nu n B + \nu C = 0$$

$$\nu n\lambda A + \{(1+\alpha)n^2 + \alpha(1-\nu)\lambda^2 - \lambda^2\phi\}B + \{n + \alpha n(\lambda^2+n^2)\}C = 0$$

~~Aux~~

$$\nu\lambda A + n\{1+\alpha[n^2+(2-\nu)\lambda^2]\}B + [1-\lambda^2\phi+\alpha(\lambda^2+n^2)^2]C = 0.$$

Determinant to be zero.

$$\underline{\lambda\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi\}\{1-\lambda^2\phi+\alpha(\lambda^2+n^2)^2\}}$$

$$+ \underline{\nu n\{n+\alpha n(\lambda^2+n^2)\}\nu\lambda} + \nu^2 n^2\lambda\{1+\alpha[n^2+(2-\nu)\lambda^2]\}$$

$$- \underline{\nu^2\lambda\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi\}} - \nu^2 n^2\lambda[1-\lambda^2\phi+\alpha(\lambda^2+n^2)^2]$$

$$- \underline{n\lambda\{1+\alpha[n^2+(2-\nu)\lambda^2]\}\{n+\alpha n(\lambda^2+n^2)\}} = 0.$$

$$(1-\nu^2)\lambda\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi\}$$

$$-\lambda^3\phi\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2\}$$

$$+\alpha(\lambda^2+n^2)^2\lambda\{n^2-\lambda^2\phi-\nu^2n^2\}$$

$$-n\lambda(1-\nu^2)\{n+\alpha n(\lambda^2+n^2)\} - n^2\lambda\alpha[n^2+(2-\nu)\lambda^2](1-\nu^2)$$

$$- \qquad + \nu^2 n^2\lambda^3\phi = 0 \ldots , = 0.$$

$$(1-\nu^2)\lambda\left\{\alpha n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi-\alpha n^2(\lambda^2+n^2)\right\} \quad 4)$$

$$-\lambda^3\phi\left\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2\right\}$$

$$+\alpha(\lambda^2+n^2)^2\lambda\left\{(1-\nu^2)n^2-\lambda^2\phi\right\}+\nu^2 n^2\lambda^3\phi$$

$$-n^2\lambda\alpha(1-\nu^2)\left[n^2+(2-\nu)\lambda^2\right]=0.$$

$$\phi\left\{\lambda^3(1-\nu^2)+\lambda^3\left[(1+\alpha)n^2+\alpha(1-\nu)\lambda^2\right]\right.$$
$$\left.+\alpha\lambda^3(\lambda^2+n^2)^2-\nu^2 n^2\lambda^3\right\}$$

$$=(1-\nu^2)\lambda\left\{\alpha n^2+\alpha(1-\nu)\lambda^2-\alpha n^2(\lambda^2+n^2)\right\}$$
$$+\alpha(\lambda^2+n^2)^2\lambda(1-\nu^2)n^2-n^2\lambda\alpha(1-\nu^2)\left[n^2+(2-\nu)\lambda^2\right]$$

$$\lambda^2\phi\left[(1-\nu^2)+(1+\alpha)n^2+\alpha(1-\nu)\lambda^2+\alpha(\lambda^2+n^2)^2-\nu^2 n^2\right]$$

$$=\alpha(1-\nu^2)\left\{n^2+(1-\nu)\lambda^2\cancel{-n^2(\lambda^2+n^2)+n^2(\lambda^2+n^2)}\right.$$
$$\left.-n^4-n^2(2-\nu)\lambda^2\right\}$$

5)

$$\phi = \frac{\alpha(1-\nu^2)\left\{n^2(1-n^2) + \lambda^2\left[(1-\nu) - n^2(2-\nu)\right]\right\}}{\lambda^2\left\{(1-\nu^2)(1+n^2) + \alpha\left[n^2 + (1-\nu)\lambda^2 + (\lambda^2+n^2)^2\right]\right\}}$$

$$P = 0.$$

$$\underline{\underline{\sigma_{cr} \approx E\left(\frac{t}{a}\right)^2}}$$

6)

$$(1-\nu^2)\cancel{p}\left\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi\right\}-\lambda^2\phi\left\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi\right\}$$

$$+\alpha\cancel{p}(\lambda^2+n^2)^2\left\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\lambda^2\phi\right\}$$

$$+\nu^2n^2\cancel{p}\left\{\alpha\left[n^2+(2-\nu)\lambda^2\right]+\lambda^2\phi-\alpha(\lambda^2+n^2)^2\right\}$$

$$-n\cancel{p}(1-\nu^2)\left\{n+\alpha n(\lambda^2+n^2)\right\}-n^2\cancel{p}\alpha\left[1+\alpha(\lambda^2+n^2)\right]\left[n^2+(2-\nu)\lambda^2\right]$$

$$=0.$$

$$(1-\nu^2)\left\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-n^2-\alpha n^2(\lambda^2+n^2)\right\}$$

$$+\alpha(\lambda^2+n^2)^2\left\{(1+\alpha)n^2+\alpha(1-\nu)\lambda^2-\nu^2n^2\right\}$$

$$+\nu^2n^2\alpha\left[n^2+(2-\nu)\lambda^2\right]-n^2\alpha\left[1+\alpha(\lambda^2+n^2)\right]\left[n^2+(2-\nu)\lambda^2\right]$$

Coeff. for $(\lambda^2\phi)$

$$-(1-\nu^2)-(1+\alpha)n^2-\alpha(1-\nu)\lambda^2$$

$$-\alpha(\lambda^2+n^2)^2+\nu^2n^2$$

Coeff. $(\lambda^2\phi)^2$ $\quad 1$

$$(\lambda^2\phi)^2-B(\lambda^2\phi)+C=0.$$

$$B = (1-\nu^2)(n^2+1) + \alpha\left\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\right\}$$

$$C = \alpha\Big[(1-\nu^2)\left\{n^2\left[1+(1-\nu)\lambda^2\right] + (1-\nu)\lambda^2 - \alpha n^2\lambda^2(\lambda^2+n^2)\right\}$$

$$+ (\lambda^2+n^2)^2\left\{(1-\nu^2)n^2 + \alpha(1-\nu)\lambda^2\right\}\Big]$$

$$B^2 - 4C = (1-\nu^2)^2(n^2+1)^2 + 2\alpha(1-\nu^2)(n^2+1)\left\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\right\}$$

$$+ \alpha^2\left\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\right\}^2$$

$$- 4\alpha\Big[(1-\nu^2)\left\{n^2\left[1+(1-\nu)\lambda^2\right] + (1-\nu)\lambda^2 + n^2(\lambda^2+n^2)^2\right\}$$

$$- 4\alpha^2\left[-(1-\nu^2)n^2\lambda^2(\lambda^2+n^2) + (\lambda^2+n^2)^2\lambda^2(1-\nu)\right]$$

III

$$B^2 - 4C = (1-\nu^2)^2(n^2+1)^2$$

$$+ 2\alpha(1-\nu^2)\Big[(n^2+1)\left\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\right\} - 2(n^2+1)(1-\nu)\lambda^2$$

$$- 2n^2 - 2n^2(\lambda^2+n^2)^2\Big]$$

$$+ \alpha^2\Big[\left\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\right\}^2 + 2\left\{(1-\nu^2)n^2\lambda^2(\lambda^2+n^2) - (1-\nu)\lambda^2(n^2+\lambda^2)^2\right\}$$

8)

$$B^2 - 4C = (1-\nu^2)^2(n^2+1)^2$$

$$+ 2\alpha(1-\nu^2)\Big[(n^2+1)\big\{(\lambda^2+n^2)(1+\lambda^2+n^2) - (2-\nu)\lambda^2\big\} - 2n^2\big\{1+(\lambda^2+n^2)^2\big\}\Big]$$

$$+ \alpha^2\Big[\big\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\big\}^2 + 2\lambda^2(\lambda^2+n^2)\big\{(1-\nu^2)n^2 - (1-\nu)(\lambda^2+n^2)\big\}\Big]$$

$$= (1-\nu^2)^2(n^2+1)^2$$

$$+ 2\alpha(1-\nu^2)\Big[(n^2-1)(\lambda^2+n^2)(1-\lambda^2-n^2) + \lambda^2\big\{\nu - n^2(2-\nu)\big\}\Big]$$

$$+ \alpha^2\Big[\big\{(\lambda^2+n^2)(1+\lambda^2+n^2) - \nu\lambda^2\big\}^2 + 2\lambda^2(\lambda^2+n^2)(1-\nu)(\nu n^2 - \lambda^2)\Big]$$

$$\approx \cancel{(1-\nu^2)^2(n^2+1)^2}$$

$$\cancel{+ 2\alpha(1-\nu^2)\big[-n^2(\lambda^2+n^2)^2 - \lambda^2 n^2(2-\nu)\big]}$$

$$\cancel{+ \alpha^2\big[\{(\lambda^2+n^2)^4\big]}$$

$$\phi = \frac{1}{2\lambda^2}\Big\{B \pm \sqrt{B^2 - 4C}\Big\}$$

If n and λ are large compared with unity, we have

$$B = n^2(1-\nu^2) + \alpha\left\{ (\lambda^2+n^2)^2 \right\}$$

$$C = \alpha\left[(1-\nu^2)\left\{ (n^2+1)(1-\nu)\lambda^2 + n^2 \right\} + (\lambda^2+n^2)^2(1-\nu^2)n^2 \right]$$

$$+\alpha^2\left[-(1-\nu^2)n^2\lambda^2(\lambda^2+n^2) + (1-\nu)\lambda^2(\lambda^2+n^2)^2 \right]$$

$$\approx \alpha\left[(1-\nu^2)\, n^2 (\lambda^2+n^2)^2 \right]$$

$$+ \alpha^2\left\{ (\lambda^2+n^2) - (1+\nu)n^2 \right\} \lambda^2(\lambda^2+n^2)(1-\nu)$$

$$\approx \alpha(1-\nu^2)n^2(\lambda^2+n^2)^2 + \alpha^2\lambda^2(1-\nu)(\lambda^2+n^2)(\lambda^2-\nu n^2)$$

$$B^2-4C = n^4(1-\nu^2)^2 + 2\alpha n^2(1-\nu^2)(\lambda^2+n^2)^2 + \alpha^2(\lambda^2+n^2)^4$$

$$- 4\alpha(1-\nu^2)n^2(\lambda^2+n^2)^2 - 4\alpha^2\lambda^2(1-\nu)(\lambda^2+n^2)(\lambda^2-\nu n^2)$$

$$\cong \left\{ n^2(1-\nu^2) - \alpha(\lambda^2+n^2)^2 \right\}^2 -$$

$$\phi \doteq \frac{1}{2\lambda^2}\; 2\alpha(\lambda^2+n^2)^2$$

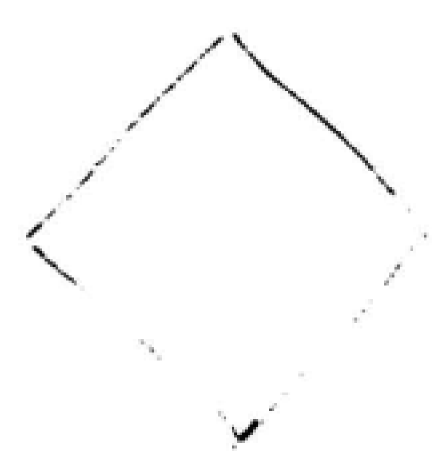
$$= \cancel{\alpha\left(\frac{\lambda^2+n^2}{\lambda^2}\right)} = \alpha\,\frac{(\lambda^2+n^2)^2}{\lambda^2}$$

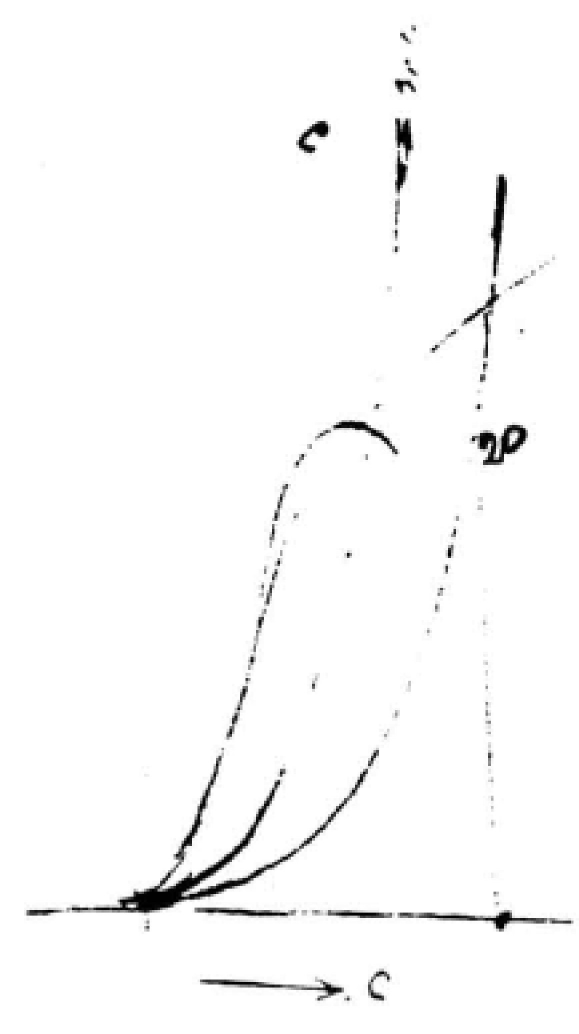
$$\sigma_{cr} = \frac{E}{12(1-\nu^2)}\left(\frac{t}{R}\right)^2 + \frac{(\lambda^2+n^2)^2}{\lambda^2}$$

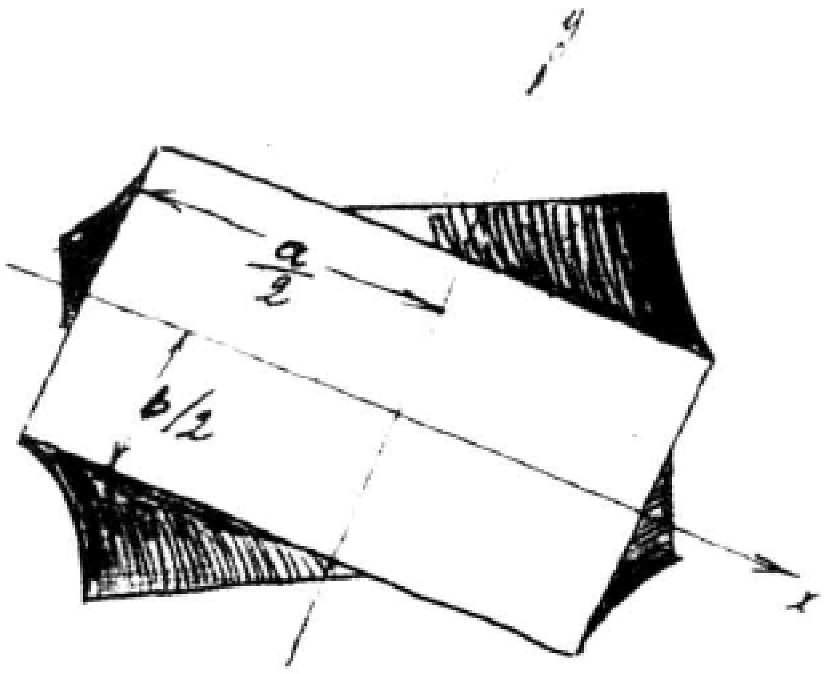
If we put $\lambda^2 = n^2$

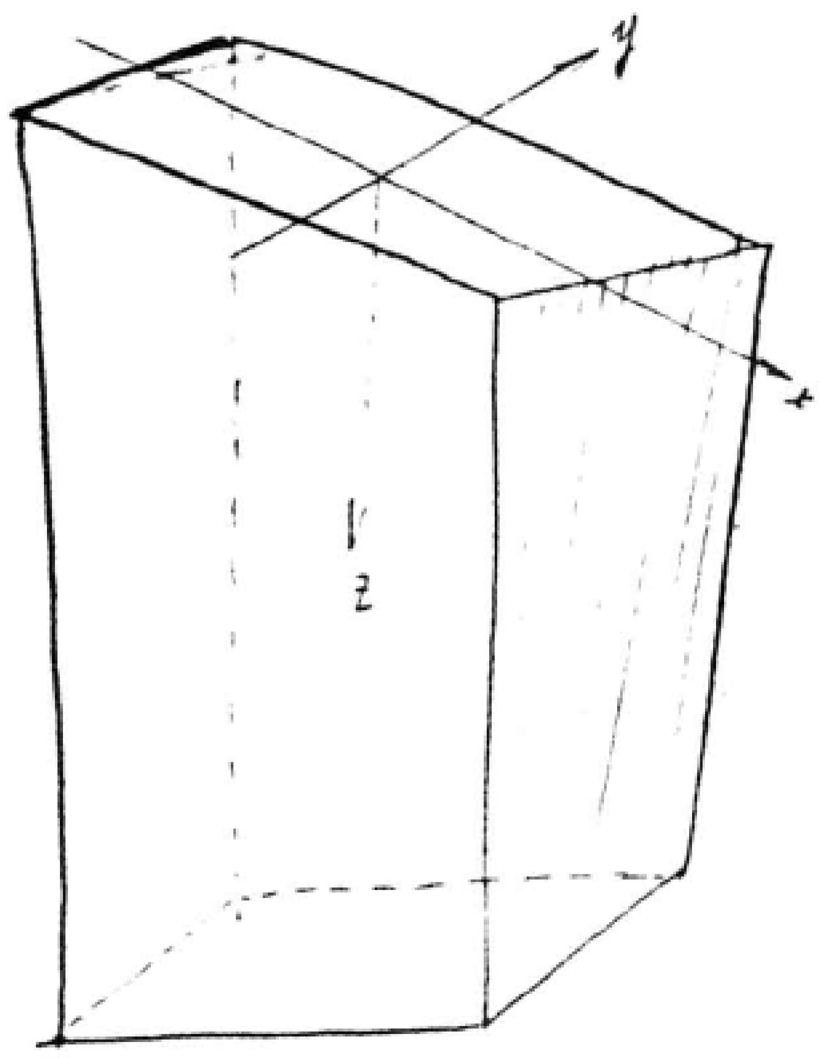
$$\sigma = \frac{E}{12(1-\nu^2)}\left(\frac{t}{R}\right)^2 4n^2 = \frac{E n^2}{3(1-\nu^2)}$$

$$= \frac{n^2}{3(1-\nu^2)} \quad E\left(\frac{t}{R}\right)^2$$









Here $w = a \cdot y$

$dF = a\,dy$

$$C_{BT} = 2\int_0^{\frac{b}{2}} w^2\,dF$$

$$= 2a^3 \int_0^{\frac{b}{2}} y^2\,dy$$

$$= \frac{2}{3}a^3\left[\,y^3\right]_0^{\frac{b}{2}} = \underline{\underline{\frac{a^3b^3}{12}}}$$

For plate of unit width $= C_{BT} = \underline{\underline{\frac{t^3}{12(1-\nu^2)}}}$

$$\begin{cases} \epsilon_1 = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \epsilon_2 = \dfrac{1}{a}\left(\dfrac{\partial v}{\partial \theta} - w\right) + \dfrac{1}{2}\dfrac{1}{a^2}\left(\dfrac{\partial w}{\partial \theta}\right)^2 \\ \varpi = \dfrac{\partial v}{\partial x} + \dfrac{1}{a}\dfrac{\partial u}{\partial \theta} + \dfrac{1}{a}\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial \theta} \end{cases}$$

If the strain energy is W, then we have

$$2W = (\lambda+2\mu)(\epsilon_1+\epsilon_2)^2 + \mu(\varpi^2 - 4\epsilon_1\epsilon_2)$$

now $$\lambda + 2\mu = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} + \frac{2E}{2(1+\sigma)}$$

$$= \frac{E}{1+\sigma}\left\{\frac{\sigma}{1-2\sigma} + 1\right\}$$

$$= \frac{E}{1+\sigma}\left\{\frac{\sigma+1-2\sigma}{1-2\sigma}\right\} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}$$

$$\mu = \frac{E}{2(1+\sigma)}$$

$$\cancel{2W = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}}$$

$$(\epsilon_1+\epsilon_2)^2 = \epsilon_1^2 + 2\epsilon_1\epsilon_2 + \epsilon_2^2$$

$$= \overset{1}{\left(\frac{\partial u}{\partial x}\right)^2} + \overset{2}{\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2} + \overset{3}{\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4} + \overset{4}{\frac{2}{a}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial \theta} - w\right)} + \overset{5}{\frac{1}{a}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial v}{\partial \theta} - w\right)}$$

$$+ \overset{6}{\frac{1}{a^2}\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial \theta}\right)^2} + \overset{7}{\frac{1}{2}\frac{1}{a^2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial \theta}\right)^2} + \overset{8}{\frac{1}{a^2}\left(\frac{\partial v}{\partial \theta} - w\right)^2} + \overset{9}{\frac{1}{a^3}\left(\frac{\partial w}{\partial \theta}\right)^2\left(\frac{\partial v}{\partial \theta} - w\right)}$$

$$+ \overset{10}{\frac{1}{4}\frac{1}{a^4}\left(\frac{\partial w}{\partial \theta}\right)^4}$$

~~13~~

$$A^2 - 4\epsilon_1\epsilon_2 = \overset{11}{\left(\frac{\partial v}{\partial x}\right)^2} + \overset{12}{\frac{1}{a^2}\left(\frac{\partial u}{\partial \theta}\right)^2} + \cancel{\frac{1}{a^2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial \theta}\right)^2}$$

$$+ \overset{13}{\frac{2}{a}\frac{\partial v}{\partial x}\frac{\partial u}{\partial \theta}} + \overset{14}{\frac{2}{a}\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}} + \overset{15}{\frac{2}{a^2}\frac{\partial u}{\partial \theta}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}}$$

$$- \overset{16}{\frac{4}{a}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial \theta} - w\right)} - \overset{17}{\frac{2}{a}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial v}{\partial \theta} - u\right)} - \overset{18}{\frac{2}{a^2}\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial \theta}\right)^2} - \cancel{\frac{1}{a^2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial \theta}\right)^2}$$

Now Put

$$u = C\cdot\alpha \sin n\theta \cos \frac{m\pi x}{l}$$

$$v = C\cdot\beta \cos n\theta \sin \frac{m\pi x}{l}$$

$$w = C \sin n\theta \sin \frac{m\pi x}{l}$$

The wave length in circumferential direction

$$\frac{2\pi}{n} a$$

The wave length in axial direction

$$2\frac{l}{m}$$

if the waves lengths are equal, then

$$\frac{2\pi a}{n} = \frac{2l}{m}$$

$$\therefore \quad m = \cancel{2} l \frac{n}{\cancel{2}\pi a} = \frac{ln}{a} \qquad \frac{ln}{a}\frac{\pi x}{l}$$

$\therefore$

$$u = C\cdot\alpha \sin n\theta \cos \frac{n\pi x}{a}$$

$$v = C\cdot\beta \cos n\theta \sin \frac{n\pi x}{a}$$

$$w = C \sin n\theta \sin \frac{n\pi x}{a}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 = C^2\alpha^2\left(\frac{n\pi}{a}\right)^2 \sin^2 n\theta \sin^2\frac{n\pi x}{a}$$ 14)

Let us find the strain energy in a cylinder of height equal to one wave length $= \frac{2a}{n}$

$$\iint\left(\frac{\partial u}{\partial x}\right)^2 d\theta\, dx = C^2\alpha^2\left(\frac{n\pi}{a}\right)^2 \int_0^{\frac{2a}{n}}\int_0^{2\pi} \sin^2 n\theta \sin^2\frac{n\pi x}{a}\, dx\, d\theta$$

$$= \pi C^2\alpha^2\left(\frac{n\pi}{a}\right)^2 \int_0^{\frac{2a}{n}} \sin^2\frac{n\pi x}{a}\, dx$$

1 $$\underline{= \pi C^2\alpha^2\left(\frac{n\pi}{a}\right)^2 \frac{a}{n} = C^2\alpha^2\pi^2\left(\frac{n\pi}{a}\right)}$$

$$\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 = -C^3\alpha\left(\frac{n\pi}{a}\right)^3 \sin^3 n\theta \sin\frac{n\pi x}{a}\cdot\cos^2\frac{n\pi x}{a}$$

2 $$\underline{\iint \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 d\theta\, dx = 0}$$

$$\left(\frac{\partial w}{\partial x}\right)^4 = C^4\left(\frac{n\pi}{a}\right)^4 \sin^4 n\theta \cos^4\frac{n\pi x}{a}$$

$$\frac{1}{4}\iint\left(\frac{\partial w}{\partial x}\right)^4 d\theta\, dx = C^4\left(\frac{n\pi}{a}\right)^4 \int_0^{\frac{2a}{n}}\int_0^{2\pi} \sin^4 n\theta \cos^4\frac{n\pi x}{a}\, d\theta\, dx$$

$$= \frac{3\pi}{1\cancel{6}} C^4\left(\frac{n\pi}{a}\right)^4 \int_0^{\frac{2a}{n}} \cos^4\left(\frac{n\pi x}{a}\right) dx$$

3 $$= \frac{9\pi}{\cancel{64}} C^4\left(\frac{n\pi}{a}\right)^4 \frac{a}{n} = \underline{\frac{9\pi^2}{\cancel{64}} C^4\left(\frac{n\pi}{a}\right)^3}$$

15)

$$\frac{2}{a}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial \theta}-w\right) = -\frac{2}{a}\cdot C\cdot\alpha \sin n\theta \sin\frac{n\pi x}{a}\left(\frac{n\pi}{a}\right)$$

$$\left\{-C\cdot\beta n \sin n\theta \sin\frac{n\pi x}{a} - C\sin n\theta\sin\frac{n\pi x}{a}\right\}$$

$$= C^2\alpha\,\frac{2}{a}\left(\frac{n\pi}{a}\right)(n\beta+1)\sin^2 n\theta\sin^2\frac{n\pi x}{a}$$

$$\frac{2}{a}\iint\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial\theta}-w\right)d\theta\,dx = C^2\cdot\alpha\,\frac{2}{a}\left(\frac{n\pi}{\cancel{a}}\right)(n\beta+1)\pi\left(\frac{\cancel{a}}{n}\right)$$

$$\underline{4 = C^2\cdot\alpha\,\frac{2\pi^2}{a}(n\beta+1)}$$

$$\underline{5 = 6 = 0}$$

$$\frac{1}{2}\frac{1}{a^2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial\theta}\right)^2 = \frac{1}{2a^2}C^4 n^2\sin^2 n\theta\cos^2 n\theta\cdot\left(\frac{n\pi}{a}\right)^2\sin^2\frac{n\pi x}{a}\cos^2\frac{n\pi x}{a}$$

$$= C^4\cdot\frac{1}{2a^2}n^2\left(\frac{n\pi}{a}\right)^2\sin^2 n\theta\cos^2 n\theta\cdot\sin^2\left(\frac{n\pi x}{a}\right)\cos^2\frac{n\pi x}{a}$$

$$\frac{1}{2a^2}\iint\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial\theta}\right)^2 d\theta\,dx = \frac{1}{16}C^4\cdot\frac{1}{2a^2}n^2\left(\frac{n\pi}{a}\right)^2\int_0^{\frac{2a}{n}}\int_0^{2\pi}\sin^2 2n\theta\,\sin^2\frac{2n\pi x}{a}\,d\theta\,dx$$

$$= \frac{1}{16}C^4\cdot\frac{1}{2a^2}n^2\left(\frac{n\pi}{a}\right)^2\pi\,\cancel{\frac{n}{a}}\,\frac{a}{n}$$

$$= \cancel{C^4\cdot\frac{\pi}{32a}\left(\frac{n\pi}{a}\right)^2}$$

$$7 = C^4\cdot\frac{1}{32}\left(\frac{n\pi}{a}\right)^3$$

16)

$$\frac{1}{a^2}\left(\frac{\partial v}{\partial \theta}-w\right)^2=\frac{1}{a^2}C^2(n\beta+1)^2\sin^2 n\theta\,\sin^2\frac{n\pi x}{a}$$

$$\frac{1}{a^2}\iint\left(\frac{\partial v}{\partial\theta}-w\right)^2 d\theta\,dx = \frac{1}{a^2}C^2(n\beta+1)^2\pi\,\frac{a}{n}$$

$$\underline{8 = C^2(n\beta+1)^2\frac{\pi}{an} = C^2\left(\beta+\frac{1}{n}\right)^2\left(\frac{n\pi}{a}\right)}$$

$$\underline{9=0}$$

$$\frac{1}{4a^4}\left(\frac{\partial w}{\partial\theta}\right)^4=\frac{1}{4a^4}C^4n^4\cos^4 n\theta\,\sin^4\frac{n\pi x}{a}$$

$$\frac{1}{4a^4}\iint\left(\frac{\partial w}{\partial\theta}\right)^4 d\theta\,dx=\frac{1}{4a^4}C^4n^4\int_0^{\frac{2a}{n}}\int_0^{2\pi}\cos^4 n\theta\,\sin^4\frac{n\pi x}{a}\,d\theta\,dx$$

$$\underline{10=\frac{1}{4a^4}C^4n^4\frac{9}{16}\pi\cdot\frac{2a}{n}=\frac{9}{64}C^4\left(\frac{n}{a}\right)^3\pi}$$

$$\left(\frac{\partial v}{\partial x}\right)^2= C^2\beta^2\left(\frac{n\pi}{a}\right)^2\cos^2 n\theta\,\cos^2\frac{n\pi x}{a}$$

$$\iint\left(\frac{\partial v}{\partial x}\right)^2 d\theta\cdot dx = C^2\beta^2\left(\frac{n\pi}{a}\right)^2\pi\cdot\frac{a}{n}$$

$$11 = C^2\beta^2\pi^2\left(\frac{n\pi}{a}\right)$$

$$\iint\frac{1}{a^2}\left(\frac{\partial u}{\partial\theta}\right)^2 d\theta\,dx=\frac{1}{a^2}C^2\alpha^2n^2\pi\cdot\frac{a}{n}=C^2\alpha^2\left(\frac{n\pi}{a}\right)=12$$

$$\underline{13=\frac{2}{\cancel{a}}C^2\alpha\beta\,\cancel{n}\cdot\left(\frac{n\pi}{a}\right)\pi\cdot\frac{\cancel{a}}{\cancel{n}}=C^2\alpha\beta\cdot 2\pi\cdot\left(\frac{n\pi}{a}\right)}$$

17)

$$14 = 15 = 0$$

$$16 = -C^2 \alpha \frac{4\pi^2}{a}(n\beta + 1)$$

$$17 = 18 = 0$$

$$2W = (\lambda + 2\mu) a \cdot t \left\{ C^2 \alpha^2 \pi^2 \left(\frac{n\pi}{a}\right) + \frac{9\pi^2}{16} C^4 \left(\frac{n\pi}{a}\right)^3 + C^2 \alpha \frac{2\pi^2}{a}(n\beta + 1) \right.$$

$$\left. + C^4 \frac{1}{32} \left(\frac{n\pi}{a}\right)^3 + C^2 \left(\beta + \frac{1}{n}\right)^2 \left(\frac{n\pi}{a}\right) + \frac{9}{64} C^4 \pi \left(\frac{n}{a}\right)^3 \right\}$$

$$+ \mu \cdot a \cdot t \left\{ C^2 \beta^2 \pi^2 \left(\frac{n\pi}{a}\right) + C^2 \alpha\beta \cdot 2\pi \left(\frac{n\pi}{a}\right) - C^2 \alpha \frac{4\pi^2}{a}(n\beta + 1) \right\}$$

$$\frac{2W}{C^2 at} = (\lambda + 2\mu) \left\{ \alpha^2 \pi^2 \left(\frac{n\pi}{a}\right) + \frac{9\pi^2}{16} \left(\frac{n\pi}{a}\right)^3 C^2 + \alpha \cdot 2\pi \left(\frac{n\pi}{a}\right)\left(\beta + \frac{1}{n}\right) \right.$$

$$\left. + \frac{1}{32} \left(\frac{n\pi}{a}\right)^3 C^2 + \left(\frac{n\pi}{a}\right)\left(\beta + \frac{1}{n}\right)^2 + \frac{9}{64}\left(\frac{n\pi}{a}\right)^3 \frac{1}{\pi^2} C^2 \right\}$$

$$+ \mu \left\{ \beta^2 \pi^2 \left(\frac{n\pi}{a}\right) + \alpha\beta \cdot 2\pi \left(\frac{n\pi}{a}\right) - \alpha\, 4\pi \left(\frac{n\pi}{a}\right)\left(\beta + \frac{1}{n}\right) \right\}$$

$$\frac{2W}{C^2 at} \frac{a}{n\pi} = (\lambda + 2\mu) \left\{ \alpha^2 \pi^2 + \frac{9\pi^2}{16} \left(\frac{n\pi}{a}\right)^2 C^2 + \alpha \cdot 2\pi \left(\beta + \frac{1}{n}\right) \right.$$

$$\left. + \frac{1}{32} \left(\frac{n\pi}{a}\right)^2 C^2 + \left(\beta + \frac{1}{n}\right)^2 + \frac{9}{64} \left(\frac{n\pi}{a}\right)^2 \frac{1}{\pi^2} C^2 \right\}$$

$$+ \mu \left\{ \beta^2 \pi^2 + \alpha\beta \cdot 2\pi - \alpha \cdot 4\pi \left(\beta + \frac{1}{n}\right) \right\}$$

$$\frac{\partial}{\partial\alpha}\mathcal{J} = (\lambda+2\mu)\left\{2\alpha\pi^2 + 2\pi\left(\beta+\frac{1}{n}\right)\right\} \qquad 18)$$

$$+\ \mu\left\{\beta\cdot 2\pi - 4\pi\left(\beta+\frac{1}{n}\right)\right\} = 0.$$

$$(\lambda+2\mu)\left\{2\pi\alpha + 2\left(\beta+\frac{1}{n}\right)\right\} + \mu\left\{2\beta - 4\left(\beta+\frac{1}{n}\right)\right\} = 0.$$

$$\text{or} \quad 2\pi(\lambda+2\mu)\alpha + 2\lambda\left(\beta+\frac{1}{n}\right) + 2\mu\beta = 0.$$

$$\cancel{2\pi(\lambda+2\mu)\alpha}$$

$$\underline{\underline{\pi(\lambda+2\mu)\alpha + (\lambda+\mu)\beta + \frac{\lambda}{n} = 0.}}$$

$$\frac{\partial}{\partial\beta}\mathcal{J} = (\lambda+2\mu)\left\{2\pi\alpha + 2\left(\beta+\frac{1}{n}\right)\right\} + \mu\left\{2\pi^2\beta + 2\pi\alpha - 4\pi\alpha\right\}$$

$$= \cancel{2\pi\lambda\alpha + 2(\lambda+2\mu)\left(\beta+\frac{1}{n}\right) + \mu 2\pi\beta -}$$

$$= \cancel{2}\pi(\lambda+\mu)\alpha + \cancel{2}(\lambda+2\mu)\left(\beta+\frac{1}{n}\right) + \cancel{2}\pi^2\mu\beta = 0$$

$$\underline{\underline{\pi(\lambda+\mu)\alpha + (\lambda+2\mu+\pi^2\mu)\beta + \frac{\lambda+2\mu}{n} = 0.}}$$

$$\left\{\pi(\lambda+2\mu)(\lambda+2\mu+\pi\mu) - \pi(\lambda+\mu)^2\right\}\beta = \frac{(\lambda+\mu)(\lambda+2\mu)}{n} - \frac{\lambda(\lambda+2\mu+\pi\mu)}{n}$$

$$\alpha = \frac{1}{n\pi}\left\{\frac{(\lambda+\mu)(\lambda+2\mu) - \lambda(\lambda+2\mu+\pi\mu)}{(\lambda+2\mu)(\lambda+2\mu+\pi\mu) - (\lambda+\mu)^2}\right\} = \frac{\cancel{1}}{n\pi}\ \cancel{\lambda+(2-\lambda)}$$

$$= \frac{1}{n\pi}\,\frac{2\mu \bullet (\pi-1)\lambda}{(\pi+2)\lambda + (3+2\pi)\mu} = \frac{1}{n\pi}\,\frac{2\mu - (\pi-1)\lambda}{(3+2\pi)\mu + (\pi+2)\lambda} = \frac{\gamma}{\pi n}$$

$$\lambda\mu \left\{ \pi(\lambda+\mu)^2 - \pi(\lambda+2\mu)(\lambda+2\mu+\pi\mu) \right\} \delta$$

$$= \frac{1}{n} \left\{ \pi(\lambda+2\mu)(\lambda+2\mu) - \pi\lambda(\lambda+\mu) \right\}$$

$$\beta = \frac{1}{n} \frac{-(4\mu+3\lambda)}{(3+2\pi)\mu + (\pi+2)\lambda} = \frac{\delta}{n}$$

$$\cancel{\frac{2W}{C^2\,t}\frac{2}{n\pi} = (\lambda+2\mu)\left\{ \frac{1}{n^2}\left[\frac{2\mu-(\pi-1)\lambda}{(3+2\pi)\mu+(\pi+2)\lambda}\right]^2 + \frac{9\pi}{16}\left(\frac{n\pi}{a}\right)^2 C^2 \right.}$$

$$+ \cancel{2\mu}$$

$$\frac{2W}{C^2\,t}\frac{1}{n\pi} = (\lambda+2\mu)\left\{ \left(\frac{\gamma}{n}\right)^2 + \frac{9}{16}\pi^2\left(\frac{n\pi}{a}\right)^2 C^2 + \frac{2\gamma}{n^2}(\delta+1) \right.$$

$$\left. + \frac{1}{32}\left(\frac{n\pi}{a}\right)^2 C^2 + \frac{1}{n^2}(\delta+1)^2 + \frac{9}{64}\left(\frac{n\pi}{a}\right)^2 \frac{1}{\pi^2} C^2 \right\}$$

$$+ \mu \left\{ \frac{\pi^2\delta^2}{n^2} + \frac{2\gamma\delta}{n^2} - \frac{4\gamma}{n^2}(\delta+1) \right\}$$

$$= (\lambda+2\mu)\left\{ \frac{1}{n^2}\left(\gamma^2 + 2\gamma\delta + 2\gamma + \delta^2 + 2\delta + 1\right) \right.$$

$$\left. + \cancel{\frac{1}{16}}\frac{1}{64} n^2 \left(\frac{c}{a}\right)^2 \left[36\pi^4 + 2\pi^2 + 1\right) \right\} + \mu\frac{1}{n^2}\left\{ \pi\delta^2 + 2\gamma\delta - 4\gamma\delta - 4\gamma \right\}$$

$$= \frac{1}{n^2}\left\{ \lambda\left(\gamma^2 + 2\gamma\delta + 2\gamma + \delta^2 + 2\delta + 1\right) + \mu\left(2\gamma^2 + 2\gamma\delta + (2+\pi)\delta^2 + 4\delta + 1\right) \right\}$$

$$+ \frac{(\lambda+2\mu)}{64} n^2 \left(\frac{c}{a}\right)^2 \left(36\pi^4 + 2\pi^2 + 1\right)$$

20)

$$\frac{1}{a^3}\left[\frac{2W}{\left(\frac{c}{a}\right)^2\left(\frac{t}{a}\right)}\,\frac{1}{n\pi}\right] = \frac{1}{n^2}\left[\lambda\left\{(\gamma+2\delta+2)\gamma + (\delta+1)^2\right\} + \mu\left\{2\gamma^2+2\gamma\delta+(2+\pi)\delta^2+4\delta+2\right\}\right]$$

$$+\ n^2\ \frac{(\lambda+2\mu)}{64}\left(\frac{c}{a}\right)^2\left(36\pi^4+2\pi^2+1\right)$$

$$\frac{1}{a^3}\left\{\frac{2W}{\pi\left(\frac{t}{a}\right)}\right\} = n\left(\frac{c}{a}\right)^2\left[\frac{\lambda\left\{(\gamma+2\delta+2)\gamma+(\delta+1)^2\right\}+\mu\left\{2\gamma^2+2\gamma\delta+(2+\pi)\delta^2+4\delta+2\right\}}{n^2}\right.$$

$$\left.+\ n^2\cdot\ (\lambda+2\mu)\left(\frac{c}{a}\right)^2\left(\frac{9}{16}\pi^4+\frac{\pi^2}{32}+\frac{1}{64}\right)\right]$$

$$\gamma = \frac{2\mu - (\pi-1)\lambda}{(3+2\pi)\mu + (\pi+2)\lambda} = \frac{\frac{1}{1+\sigma} - (\pi-1)\frac{\sigma}{(1+\sigma)(1-2\sigma)}}{(3+2\pi)\frac{1}{2(1+\sigma)} + (\pi+2)\frac{\sigma}{(1+\sigma)(1-2\sigma)}}$$

$$= \frac{1 - \frac{(\pi-1)\sigma}{1-2\sigma}}{\frac{3+2\pi}{2} + \frac{(\pi+2)\sigma}{1-2\sigma}} = \frac{(1-2\sigma) - (\pi-1)\sigma}{(1.5+\pi)(1-2\sigma) + (\pi+2)\sigma}$$

$$= \frac{1 - (\pi+1)\sigma}{(1.5+\pi) - (\pi+1)\sigma}$$

$$\delta = \frac{-(4\mu+3\lambda)}{(3+2\pi)\mu + (\pi+2)\lambda} = - \frac{\frac{2}{1+\sigma} + \frac{3\sigma}{(1+\sigma)(1-2\sigma)}}{\frac{3+2\pi}{2(1+\sigma)} + \frac{(\pi+2)\sigma}{(1+\sigma)(1-2\sigma)}}$$

$$= - \frac{2(1-2\sigma) + 3\sigma}{(1.5+\pi) - (\pi+1)\sigma} = - \frac{2-\sigma}{(1.5+\pi) - (\pi+1)\sigma}$$

With $\sigma = 0.3000$

$$\gamma = \frac{1 - 4.1416 \times 0.3000}{4.6415 - 4.1416 \times 0.3000} = - \frac{0.24248}{3.3991}$$

$$= -0.0714$$

$$\delta = - \frac{1.7}{3.3991} = -0.5000$$

$$\lambda = E \cdot \frac{\sigma}{(1+\sigma)(1-2\sigma)} = E \cdot \frac{0.3}{1.3 \times 0.4} = E \cdot 0.577$$

$$\begin{array}{r} 0.577 \\ 0.770 \\ \hline 1.347 \end{array}$$

$$\mu = E \frac{1}{2(1+\sigma)} = E \cdot \frac{1}{2.6} = E \cdot 0.3850$$

$$\lambda + 2\mu = E \cdot 1.347$$

$$\lambda\left\{(\gamma+2\delta+2)\gamma+(\delta+1)^2\right\} = E\cdot 0.577\left\{-(2-0.0714-1)0.0714+0.25\right\} \quad 22)$$

$$= E\ 0.577\left\{0.25-0.0663\right\} = 0.1060\ E$$

$$\mu\left\{2\gamma^2+2\gamma\delta+(2+\pi)\delta^2+4\delta+2\right\} = E\cdot 0.3850\left\{2\times 0.0714^2+0.0714\right.$$

$$\left.+\ 5.1416\times 0.25\right\} = E\cdot 0.3850\left\{0.0816+1.285\right\} = 0.526\ E$$

$$(\lambda+2\mu)\left(\frac{9}{16}\pi^4+\frac{\pi^2}{32}+\frac{1}{64}\right) = E\cdot 1.347\left(\frac{9}{16}\times 9.8696^2+\frac{9.8696}{32}+\frac{1}{64}\right)$$

$$= E\cdot 1.347\left(54.8+0.308+0.016\right) = E\cdot 1.347\times 55.1$$

$$= 74.3\ E$$

$$\boxed{\Psi = \frac{1}{a^3}\left\{\frac{2W}{\pi\left(\frac{?}{a}\right)}\right\} = E\,n\left(\frac{c}{a}\right)^2\left\{\frac{0.632}{n^2}+74.3\ n^2\left(\frac{c}{a}\right)^2\right\}} \quad ?$$

If $n = 10$,

$$\Psi = E\cdot 10\left(\frac{c}{a}\right)^2\left\{0.00632+0.00743\times\left(\frac{1000\,c}{a}\right)^2\right\}$$

$$= E\ 0.0632\left(\frac{c}{a}\right)^2\left\{1+1.175\times\left(\frac{1000\,c}{a}\right)^2\right\}$$

$$\boxed{\Psi = \frac{E}{10^6}\cdot 0.0632\left(\frac{1000c}{a}\right)^2\left\{1+1.175\times\left(\frac{1000\,c}{a}\right)^2\right\}}$$

$$G_1 = -D\left\{k_1 + \sigma k_2 + \frac{1}{R_2'}(\epsilon_1 + \sigma\epsilon_2)\right\}$$

$$G_2 = -D\left\{k_2 + \sigma k_1 + \frac{1}{R_1'}(\epsilon_2 + \sigma\epsilon_1)\right\}$$

$$H_1 = D(1-\sigma)\left(\tau + \frac{1}{2}\frac{\varpi}{R_2'}\right), \qquad H_2 = -D(1-\sigma)\left(\tau + \frac{1}{2}\frac{\varpi}{R_1'}\right)$$

$$2W = D\Big\{k_1\left[k_1 + \sigma k_2 + \frac{1}{R_2'}(\epsilon_1 + \sigma\epsilon_2)\right] + k_2\left[k_2 + \sigma k_1 + \frac{1}{R_1'}(\epsilon_2 + \sigma\epsilon_1)\right]$$

$$+ (1-\sigma)\left[\tau\left(\tau + \frac{1}{2}\frac{\varpi}{R_2'}\right) + \tau\left(\tau + \frac{1}{2}\frac{\varpi}{R_1'}\right)\right]\Big\}$$

$$= D\Big\{(k_1^2 + k_2^2) + 2\sigma k_1 k_2 + \left(\frac{k_1}{R_2'} + \frac{\sigma k_2}{R_1'}\right)\epsilon_1 + \left(\frac{\sigma k_1}{R_2'} + \frac{k_2}{R_1'}\right)\epsilon_2$$

$$+ (1-\sigma)\left[2\tau^2 + \frac{1}{2}\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)\varpi\tau\right]\Big\}$$

$$= D\Big\{(k_1 + k_2)^2 + \left(\frac{k_1}{R_2'} + \frac{\sigma k_2}{R_1'}\right)\epsilon_1 + \left(\frac{\sigma k_1}{R_2'} + \frac{k_2}{R_1'}\right)\epsilon_2$$

$$+ (1-\sigma)\left[2\tau^2 - 2k_1k_2 + \frac{1}{2}\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)\varpi\tau\right]\Big\}$$

now $\quad k_1 = \frac{\partial^2 w}{\partial x^2}, \qquad k_2 = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial v}{\partial\theta}\right) \qquad \tau = \frac{1}{a}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial\theta} + v\right)$

$$\frac{1}{R_1'} = \frac{\partial^2 w}{\partial x^2} \qquad \frac{1}{R_2'} = \frac{1}{a} + \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial v}{\partial\theta}\right)$$

$$\epsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_2 = \frac{1}{a}\left(\frac{\partial v}{\partial\theta} - w\right) + \frac{1}{2}\frac{1}{a^2}\left(\frac{\partial w}{\partial\theta}\right)^2$$

$$\varpi = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial\theta} + \frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial\theta}$$

$$(k_1+k_2)^2 = \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{a^2\partial\theta^2} + \frac{1}{a^2}\frac{\partial v}{\partial\theta}\right)^2$$

$$= \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{a^2\partial\theta^2}\right)^2 + \frac{1}{a^4}\left(\frac{\partial v}{\partial\theta}\right)^2 + 2\cdot\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{a^2\partial\theta^2} + \frac{2}{a^2}\frac{\partial v}{\partial\theta}\frac{\partial^2 w}{\partial x^2}$$

$$+ \frac{2}{a^4}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial v}{\partial\theta}$$

$$\iint\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx\,d\theta = C^2\left(\frac{n\pi}{a}\right)^4 \iint \sin^2 n\theta \sin^2\frac{n\pi x}{a}\,dx\,d\theta$$

$$1 = C^2\left(\frac{n\pi}{a}\right)^4 \pi\frac{a}{n} = C^2\pi^2\left(\frac{n\pi}{a}\right)^3$$

$$2 = \iint\frac{1}{a^4}\left(\frac{\partial^2 w}{\partial\theta^2}\right)^2 dx\,d\theta = C^2\left(\frac{n}{a}\right)^4\pi\frac{a}{n} = C^2\frac{1}{\pi^3}\left(\frac{n\pi}{a}\right)^3$$

$$3 = \iint\frac{1}{a^4}\left(\frac{\partial v}{\partial\theta}\right)^2 dx\,d\theta = \frac{1}{a^4}C^2\beta^2 n^2\pi\frac{a}{n} = C^2\beta^2\frac{1}{\pi^3 n^2}\left(\frac{n\pi}{a}\right)^3$$

$$4 = \iint\frac{2}{a^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial\theta^2}\,dx\,d\theta = \frac{2}{a^2}C^2\left(\frac{n\pi}{a}\right)^2 n^2\ \pi\cdot\frac{a}{n} = C^2\left(\frac{n\pi}{a}\right)^3 2$$

$$5 = \iint\frac{2}{a^2}\frac{\partial v}{\partial\theta}\frac{\partial^2 w}{\partial x^2} = \frac{2}{a^2}C^2\beta\left(\frac{n\pi}{a}\right)^2 n\cdot\pi\frac{a}{n} = C^2\beta\left(\frac{n\pi}{a}\right)^3\frac{2}{n}$$

$$6 = \iint\frac{2}{a^4}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial v}{\partial\theta} = \frac{2}{a^4}C^2\beta\ \cancel{\left(\frac{n\pi}{a}\right)^2}\ n^3\pi\frac{a}{n} = C^2\beta\left(\frac{n\pi}{a}\right)^3\frac{2}{n\pi^3}$$

$$\left(\frac{k_1}{R_2'} + \frac{\sigma k_2}{R_1'}\right) t_1 = \left[\frac{\partial^2 w}{\partial x^2}\left(\frac{1}{a} + \frac{\partial^2 w}{a^2 \partial \theta^2} + \frac{\partial v}{a^2 \partial \theta}\right) + \frac{\sigma}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta}\right)\frac{\partial^2 w}{\partial x^2}\right]\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$= \left\{(1+\sigma)\frac{\partial^2 w}{\partial x^2}\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta}\right) + \frac{1}{a}\frac{\partial^2 w}{\partial x^2}\right\}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$

the effect terms are $= \overset{7}{\dfrac{(1+\sigma)}{2a^2}\left(\dfrac{\partial w}{\partial x}\right)^2 \dfrac{\partial^2 w}{\partial x^2}\left(\dfrac{\partial^2 w}{\partial \theta^2} + \dfrac{\partial v}{\partial \theta}\right)} + \overset{8}{\dfrac{1}{a}\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial u}{\partial x}}$

$$\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta} = -C n^2 \sin n\theta \sin\frac{n\pi x}{a} - C\beta n \sin n\theta \sin\frac{n\pi x}{a}$$

$$= -C\left(n^2 + n\beta\right)\sin n\theta \sin\frac{n\pi x}{a}$$

$$\iint \left(\frac{1+\sigma}{2a^2}\right)\left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta}\right) d\theta\, dx$$

$$= \frac{1+\sigma}{2a^2}\left(\frac{n\pi}{a}\right)^2\left(\frac{n\pi}{a}\right)^2 C^4\left(n^2 + n\beta\right)\iint \sin^4 n\theta \sin^2\frac{n\pi x}{a}\cos^2\frac{n\pi x}{a}\, d\theta\, dx$$

$$= \frac{1+\sigma}{2a^2}\left(\frac{n\pi}{a}\right)^4\left(n^2+n\beta\right) C^4 \frac{3}{4}\pi \cdot \frac{1}{4}\frac{a}{n}$$

$$= \cancel{\frac{3(1+\sigma)}{32}\left(n^2+n\beta\right)}$$

$$7 = \frac{3(1+\sigma)}{32}\frac{(n+\beta)}{n}\left(\frac{n\pi}{a}\right)^5 C^4 = \frac{3(1+\sigma)}{32}\left(1+\frac{\beta}{n}\right)\left(\frac{n\pi}{a}\right)^5 C^4$$

$$\iint \frac{1}{a}\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial x}\, dx\, d\theta = \frac{1}{a}\left(\frac{n\pi}{a}\right)^2\left(\frac{n\pi}{a}\right) C^2 \alpha\, \pi \cdot \frac{a}{n}$$

$$8 = C^2 \alpha\left(\frac{n\pi}{a}\right)^3\frac{\pi}{n}$$

26)

$$\left(\frac{\sigma k_1}{R_2'}+\frac{k_2}{R_1'}\right)t_1=\left\{(1+\sigma)\frac{1}{a^2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial v}{\partial\theta}\right)+\frac{\sigma}{a}\frac{\partial^2 w}{\partial x^2}\right\}\left\{\frac{1}{a}\left(\frac{\partial v}{\partial\theta}-w\right)+\frac{1}{2a^2}\left(\frac{\partial w}{\partial\theta}\right)^2\right\}$$

The effective terms are

$$\overset{9}{\frac{(1+\sigma)}{2a^4}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial v}{\partial\theta}\right)\left(\frac{\partial w}{\partial\theta}\right)^2}+\overset{10}{\frac{\sigma}{a^2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial v}{\partial\theta}-w\right)}$$

$$\iint\frac{(1+\sigma)}{2a^4}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial\theta}\right)^2\left(\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial v}{\partial\theta}\right)dx\,d\theta$$

$$=\frac{(1+\sigma)}{2a^4}C^4\cdot\left(\frac{n\pi}{a}\right)^2 n^2(n^2+n\beta)\iint\sin^2 n\theta\cos^2 n\theta\sin^4\frac{n\pi x}{a}\,dx\,d\theta$$

$$9=\frac{(1+\sigma)}{2a^4}C^4\cdot\left(\frac{n\pi}{a}\right)^2 n^2(n^2+n\beta)\,\frac{\pi}{4}\,\frac{3}{4}\,\frac{a}{n}=\frac{3}{32}(1+\sigma)C^4\left(\frac{n\pi}{a}\right)^5\left(1+\frac{\beta}{n}\right)\frac{1}{\pi^2}$$

$$\iint\frac{\sigma}{a^2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial v}{\partial\theta}-w\right)dx\,d\theta$$

$$=\frac{\sigma}{a^2}C^2\left(\frac{n\pi}{a}\right)^2(n\beta+1)\iint\sin^2 n\theta\sin^2\frac{n\pi x}{a}\,dx\,d\theta$$

$$10=\frac{\sigma}{a^2}C^2\left(\frac{n\pi}{a}\right)^2(n\beta+1)\,\pi\cdot\frac{a}{n}=C^2\cdot\sigma\left(\beta+\frac{1}{n}\right)\left(\frac{n\pi}{a}\right)^3\frac{1}{n}$$

$$2\tau^2=2\left\{\frac{1}{a}\frac{\partial^2 w}{\partial x\partial\theta}+\frac{1}{a}\frac{\partial v}{\partial x}\right\}^2=\frac{2}{a^2}\left\{\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\partial v}{\partial x}\right\}^2$$

$$=\frac{2}{a^2}C^2(n+\beta)^2\left(\frac{n\pi}{a}\right)^2\cos^2 n\theta\cos^2\frac{n\pi x}{a}$$

$$11\quad\int 2\tau^2\,d\theta\,dx=\frac{2}{a^4}C^2(n+\beta)^2\left(\frac{n\pi}{a}\right)^2\pi.\,\frac{a}{n}=2C^2\left(1+\frac{\beta}{n}\right)^2\left(\frac{n\pi}{a}\right)^3$$

$$-2k_1k_2 = -\frac{2}{a^2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)$$

$$= -\frac{2}{a^2}C^2\left(\frac{n\pi}{a}\right)^2 \sin n\theta \sin\frac{n\pi x}{a}\left(n^2+n\beta\right)\sin n\theta \sin\frac{n\pi x}{a}$$

$$I_2 = -\frac{2}{a^2}C^2\left(\frac{n\pi}{a}\right)^2\left(n^2+n\beta\right)\pi\cdot\frac{\ell}{n} = -C^2\left(\frac{n\pi}{a}\right)^3 2\left(1+\frac{\beta}{n}\right)$$

$$\frac{1}{2}\left(\frac{1}{R_1'}+\frac{1}{R_2'}\right)\Delta\tau = \frac{1}{2}\left[\frac{\partial^2 w}{\partial x^2}+\frac{1}{a}+\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)\right]\left[\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial \theta}+\frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\right]$$

$$\frac{1}{a}\left[\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\partial v}{\partial x}\right]$$

The effective terms are

$$\frac{1}{2a^2}\left[\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial \theta}+\cancel{\frac{1}{a}}\right]\left[\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\partial v}{\partial x}\right]$$

$$\frac{1}{2a^2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\left[\frac{\partial^2 w}{\partial x^2}+\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)\right]\left[\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\partial v}{\partial x}\right]$$

$$\iint\frac{1}{2a^2}\left(\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial \theta}\right)d\theta\, dx$$

$$= \frac{1}{2a^2}C\left[n\cdot\left(\frac{n\pi}{a}\right)+\frac{n\pi}{a}\beta\right]\iint \cos^2 n\theta \cos^2\frac{n\pi x}{a}\cdot C\left[\left(\frac{n\pi}{a}\right)\beta+\frac{\pi}{a}\alpha\right]d\theta\, dx$$

$$= \frac{1}{2a^2}C^2\left(\frac{n\pi}{a}\right)^2\left[n+\beta\right]\left[\beta+\frac{\alpha}{\pi}\right]\pi\cdot\frac{\ell}{n}$$

$$I_3 = \frac{1}{2}C^2\left(\frac{n\pi}{a}\right)^3\left(1+\frac{\beta}{n}\right)\left(\frac{\beta}{n}+\frac{\alpha}{n\pi}\right)$$

$$\iint \frac{1}{2a^2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\left[\frac{\partial^2 w}{\partial x^2}+\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right)\right]\left[\frac{\partial^2 w}{\partial x\,\partial \theta}+\frac{\partial v}{\partial x}\right]dx\,d\theta$$

$$= \frac{1}{2a^2}\left(\frac{n\pi}{a}\right)n\left[-\left(\frac{n\pi}{a}\right)^2-\frac{1}{a^2}\left(n^2+n\beta\right)\right]\left[\left(\frac{n\pi}{a}\right)n+\beta\cdot\left(\frac{n\pi}{a}\right)\right]\iint \sin^2 n\theta\cos^2 n\theta\,\sin^2\frac{n\pi x}{a}\cos^2\frac{n\pi x}{a}\,dx\,d\theta$$

$$= -\frac{1}{2a^2}\left(\frac{n\pi}{a}\right)^4 n\left[1+\frac{1}{\pi^2}\left(1+\frac{\beta}{n}\right)\right]\left[n+\beta\right]\frac{1}{4}\cdot\frac{1}{4}\frac{a}{n}\pi\ C^4$$

$$\underline{V = -\frac{1}{32}\left(\frac{n\pi}{a}\right)^5\left[1+\frac{1}{\pi^2}\left(1+\frac{\beta}{n}\right)\right]\left[1+\frac{\beta}{n}\right]C^4}$$

$$\frac{2W}{a} = D\Big\{C^2\pi^2\left(\frac{n\pi}{a}\right)^3+C^2\frac{1}{\pi^2}\left(\frac{n\pi}{a}\right)^3+C^2\beta^2\left(\frac{n\pi}{a}\right)^3\frac{1}{n^2\pi^2}$$

$$+C^2\,2\left(\frac{n\pi}{a}\right)^3+C^2\beta\left(\frac{n\pi}{a}\right)^3\frac{2}{n}+C^2\beta\left(\frac{n\pi}{a}\right)^3\frac{2}{n\pi^2}$$

$$+C^4\frac{3(1+\sigma)}{32}\left(1+\frac{\beta}{n}\right)\left(\frac{n\pi}{a}\right)^5+C^2\alpha\left(\frac{n\pi}{a}\right)^3\frac{\pi}{n}+\frac{3(1+\sigma)}{32}C^4\left(\frac{n\pi}{a}\right)^5\left(1+\frac{\beta}{n}\right)\frac{1}{\pi^2}$$

$$+C^2\sigma\left(\beta+\frac{1}{n}\right)\left(\frac{n\pi}{a}\right)^3\frac{1}{n}+\cancel{2W}$$

$$(1-\sigma)\Big[2C^2\left(1+\frac{\beta}{n}\right)^2\left(\frac{n\pi}{a}\right)^3-2C^2\left(1+\frac{\beta}{n}\right)\left(\frac{n\pi}{a}\right)^3+\frac{1}{2}C^2\left(\frac{n\pi}{a}\right)^3\left(1+\frac{\beta}{n}\right)\left(\frac{\beta}{n}+\frac{a}{n\pi}\right)$$

$$-C^4\frac{1}{32}\left(\frac{n\pi}{a}\right)^5\left[1+\frac{\beta}{n}\right]\left[1+\frac{1}{\pi^2}\left(1+\frac{\beta}{n}\right)\right]\Big]\Big\}$$

$$\frac{2W}{a} = C^2\left(\frac{n\pi}{a}\right)^3 D\left\{ \pi^2 + \frac{1}{\pi^2} + \frac{1}{\pi^2}\left(\frac{\beta}{n}\right)^2 + 2 + 2\left(\frac{\beta}{n}\right) + 2\left(\frac{\beta}{n}\right)\frac{1}{\pi^2} + \pi\left(\frac{\alpha}{n}\right)\right.$$

$$+ \sigma\left(\frac{\beta}{n} + \frac{1}{n^2}\right) + C^2\left[1 + \frac{1}{\pi^2}\right]\frac{3(1+\sigma)}{32}\left(\frac{n\pi}{a}\right)^2\left(1 + \frac{\beta}{n}\right)\Big\}^{1}$$

$$+ (1-\sigma)\left[2\left(1+\frac{\beta}{n}\right)^2 - 2\left(1+\frac{\beta}{n}\right) + \frac{1}{2}\left(1+\frac{\beta}{n}\right)\left(\frac{\beta}{n} + \frac{\alpha}{n\pi}\right)\right.$$

$$\left. - C^2\frac{1}{32}\left(\frac{n\pi}{a}\right)^2\left[1+\frac{\beta}{n}\right]\left[1 + \frac{1}{\pi^2}\left(1+\frac{\beta}{n}\right)\right]\right\}.$$

$$\cancel{\frac{2W}{a} = C^2(n\pi)^3 E \cdot \frac{\left(\frac{t}{a}\right)^3}{(1-\sigma^2)\,12}}$$

$$\frac{2W}{a^3} = \frac{\left(\frac{C}{a}\right)^2 (n\pi)^3 E}{12(1-\sigma^2)}\left(\frac{t}{a}\right)^3\left\{\left(\pi^2 + \frac{1}{\pi^2} + 2\right) + 2\left(1+\frac{1}{\pi^2}\right)\left(\frac{\beta}{n}\right) + \frac{1}{\pi^2}\left(\frac{\beta}{n}\right)^2\right.$$

$$+ \pi\left(\frac{\alpha}{n}\right) + \sigma\left(\frac{\beta}{n} + \frac{1}{n^2}\right) + (1-\sigma)\left(1+\frac{\beta}{n}\right)\left[2\left(\cancel{1}+\frac{\beta}{n}\right) - \cancel{2} + \frac{1}{2}\left(\frac{\beta}{n} + \frac{\alpha}{n}\frac{1}{\pi}\right)\right]$$

$$\left. + \left(\frac{C}{a}\right)^2\frac{(n\pi)^2\left(1+\frac{\beta}{n}\right)}{32}\left[3(1+\sigma)\left(1+\frac{1}{\pi^2}\right) - \left\{1 + \frac{1}{\pi^2}\left(1+\frac{\beta}{n}\right)\right\}(1-\sigma)\right]\right\}$$

$$\cancel{= \frac{E(n\pi)^3}{12(1-\sigma^2)}\left(\frac{C}{a}\right)^2\left(\frac{t}{a}\right)^3\left\{\frac{\sigma}{n^2} + \left(\pi + \frac{1}{\pi}\right)^2 + 2\left(1 + \frac{1}{\pi^2} + \frac{\sigma}{2}\right)\left(\frac{\beta}{n}\right) + \pi\left(\frac{\alpha}{n}\right)\right.}$$

$$\frac{2W}{a^3} = \frac{\left(\frac{C}{a}\right)^2(n\pi)^3 E}{12(1-\sigma^2)}\left(\frac{t}{a}\right)^3\left[\left\{\left[\frac{\sigma}{n^2} + \left(\pi + \frac{1}{\pi}\right)^2\right] + \left(\frac{2}{\pi^2} + \frac{9}{2} - \frac{3\sigma}{2}\right)\left(\frac{\beta}{n}\right) + \left(\pi + \frac{1-\sigma}{2\pi}\right)\left(\frac{\alpha}{n}\right)\right.\right.$$

$$\left. + \left(\frac{1}{\pi^2} + \frac{5(1-\sigma)}{2}\right)\left(\frac{\beta}{n}\right)^2 + \frac{1-\sigma}{2\pi}\left(\frac{\beta}{n}\right)\left(\frac{\alpha}{n}\right)\right\}$$

$$\left. + \left(\frac{C}{a}\right)^2\frac{(n\pi)^2\left(1+\frac{\beta}{n}\right)}{32}\left\{2\left(1+\frac{1}{\pi^2}\right)(1+2\sigma) - \frac{(1-\sigma)}{\pi^2}\left(\frac{\beta}{n}\right)\right\}\right]$$

Extensional Strain Energy p. 17

$$\frac{2W}{a^3} = E\left(\frac{c}{a}\right)^2\left(\frac{t}{a}\right)(n\pi)^3\left[\frac{(\lambda+2\mu)}{E}\left\{\left(\frac{\alpha}{n}\right)^2 + \frac{9\pi^2}{16}\left(\frac{c}{a}\right)^2 + \frac{2}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}+\frac{1}{n^2}\right)\right.\right.$$

$$\left. + \frac{1}{32}\left(\frac{c}{a}\right)^2 + \frac{1}{\pi^2}\left(\frac{\beta}{n}+\frac{1}{n^2}\right)^2 + \frac{9}{64}\frac{1}{\pi^2}\left(\frac{c}{a}\right)^2\right\}$$

$$\left. + \frac{\mu}{E}\left\{\left(\frac{\beta}{n}\right)^2 + \frac{2}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}\right) - \frac{4}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}+\frac{1}{n^2}\right)\right\}\right]$$

$$= E\left(\frac{t}{a}\right)\left(\frac{c}{a}\right)^2(n\pi)^3\left[\frac{\lambda+2\mu}{E}\left\{\left(\frac{\alpha}{n}\right)^2 + \frac{2}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}+\frac{1}{n^2}\right) + \frac{1}{\pi^2}\left(\frac{\beta}{n}+\frac{1}{n^2}\right)^2\right\}\right.$$

$$+ \frac{\mu}{E}\left\{\left(\frac{\beta}{n}\right)^2 + \frac{2}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}\right) - \frac{4}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}+\frac{1}{n^2}\right)\right\}$$

$$\left. + \frac{(\lambda+2\mu)}{32E}\left(\frac{c}{a}\right)^2\left\{\overset{1+\pi^2}{1} + \frac{9}{2\pi^2}\right\}\right]$$

$$= \frac{E\left(\frac{t}{a}\right)\left(\frac{c}{a}\right)^2(n\pi)^3}{1-\sigma^2}\left[\frac{(1-\sigma)^2}{(1-2\sigma)}\left\{\left(\frac{\alpha}{n}\right)^2 + \frac{2}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}+\frac{1}{n^2}\right) + \frac{1}{\pi^2}\left(\frac{\beta}{n}+\frac{1}{n^2}\right)^2\right\}\right.$$

$$+ \frac{(1-\sigma)}{2}\left\{\left(\frac{\beta}{n}\right)^2 + \frac{2}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}\right) - \frac{4}{\pi}\left(\frac{\alpha}{n}\right)\left(\frac{\beta}{n}+\frac{1}{n^2}\right)\right\}$$

$$\left. + \frac{(1-\sigma)^2}{(1-2\sigma)}\frac{1}{32}\left(\frac{c}{a}\right)^2\left\{\overset{18\pi^2}{1} + \frac{9}{2\pi^2}\right\}\right]$$

Putting $\left(\frac{\alpha}{n}\right) = f$, $\left(\frac{\beta}{n}\right) = g$

Total Strain energy

$$\frac{a^2 W}{a^3} = \frac{E\left(\frac{t}{a}\right)\left(\frac{C}{a}\right)^2 (n\pi)^3}{1-\sigma^2}\Bigg[\frac{(1-\sigma)^2}{(1-2\sigma)}\left\{(\cancel{f})^2 + \frac{2}{\pi}\cancel{f}\left(g+\frac{1}{\pi^2}\right) + \frac{1}{\pi^2}\left(g+\frac{1}{\pi^2}\right)^2\right\}$$

$$+ \frac{1-\sigma}{2}\left\{g^2 + \frac{2}{\pi}fg - \frac{4}{\pi}f\left(g+\frac{1}{\pi^2}\right)\right\} + \frac{(1-\sigma)^2}{(1-2\sigma)}\frac{1}{32}\left(\frac{C}{a}\right)^2\left\{\overset{1+\pi^2}{\cancel{1}} + \frac{9}{2\pi^2}\right\}$$

$$+ \frac{\left(\frac{t}{a}\right)^2}{12}\left\{\left[\frac{\sigma}{\pi^2} + \left(\pi+\frac{1}{\pi}\right)^2\right] + \left(\frac{2}{\pi^2}+\frac{9}{2}-\frac{3\sigma}{2}\right)g + \left(\pi + \frac{1-\sigma}{2\pi}\right)f + \left(\frac{1}{\pi^2}+\frac{5(1-\sigma)}{2}\right)g^2\right.$$

$$\left. + \frac{1-\sigma}{2\pi}fg\right\} + \left(\frac{C}{a}\right)^2\frac{(n\pi)^2\left(1+\overset{9}{\cancel{\pi^2}}\right)}{32}\frac{\left(\frac{t}{a}\right)^2}{12}\left\{2\left(1+\frac{1}{\pi^2}\right)(1+2\sigma) - \frac{(1-\sigma)}{\pi^2}g\right\}\Bigg]$$

$$(\epsilon_1+\epsilon_2-e)^2 = \left[\frac{\partial u}{\partial x} - e + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{a}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{1}{2}\frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]^2$$

The effective terms are $\quad -e\left[\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right] + e^2$

$$-4\epsilon_1\epsilon_2 = -4\left[\frac{\partial u}{\partial x} - e + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{1}{a}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{1}{2}\frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]$$

$$= +4e\,\frac{1}{2}\frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$(\lambda+2\mu)\left\{e^2 - e\left[\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]\right\} + \mu \cdot 2e\,\frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$= (\lambda+2\mu)e^2 - e\left\{(\lambda+2\mu)\left(\frac{\partial w}{\partial x}\right)^2 + \lambda\frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right\}$$

$$\iint\left(\frac{\partial w}{\partial x}\right)^2 dx\,d\theta = C^2\left(\frac{n\pi}{a}\right)^2\pi\frac{a}{n} = C^2\pi^2\left(\frac{n\pi}{a}\right)$$

$$\iint\frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2 dx\,d\theta = C^2\left(\frac{n}{a}\right)^2\pi\frac{a}{n} = C^2\left(\frac{n\pi}{a}\right)$$

$$\iint dx\,d\theta = 2\pi\cdot\frac{2a}{n} = 4\left(\frac{n\pi}{a}\right)\left(\frac{a}{n}\right)^2$$

32)

$$\frac{2W}{a} = t\ (\lambda+2\mu)e^2 4\left(\frac{n\pi}{a}\right)\left(\frac{a}{n}\right)^2 - te\left\{ (\lambda+2\mu)\, c^2 \pi^2 \left(\frac{n\pi}{a}\right) + \lambda c^2 \frac{n\pi}{a} \right\}$$

$$\cancel{\frac{2W}{a^3} = \left(\frac{t}{a}\right)(\lambda+2\mu)\left(\frac{e}{a}\right)^2 4\left(\frac{n\pi}{a}\right) - \left(\frac{t}{a}\right)e\left\{ (\lambda+2\mu)\left(\frac{c}{a}\right)^2 \pi^2 \left(\frac{n\pi}{a}\right) + \lambda\left(\frac{c}{a}\right)^2 \frac{n\pi}{a} \right\}}$$

$$* \quad \cancel{\frac{2W}{a^3} = \left(\frac{t}{a}\right)(\lambda+2\mu)}$$

$$\frac{2W}{a^3} = \left(\frac{t}{a}\right)\left(\frac{n\pi}{a}\right)\left\{\frac{4(\lambda+2\mu)e^2 a}{n^2} - ae\left[(\lambda+2\mu)\pi^2 + \lambda\right]\left(\frac{c}{a}\right)^2\right\}$$

$$= \left(\frac{t}{a}\right)(n\pi)^3 \left\{ \frac{4(\lambda+2\mu)e\left(\frac{e}{a}\right)}{n^4\pi^2} - \frac{e}{n^2\pi^2}\left[(\lambda+2\mu)\pi^2 + \lambda\right]\left(\frac{c}{a}\right)^2 \right\}$$

$$= \frac{E\left(\frac{t}{a}\right)(n\pi)^3}{1-\sigma^2}\left\{ \frac{4e^2}{n^4\pi^2}\ \frac{(1-\sigma)^2}{(1-2\sigma)} - \frac{e}{n^2\pi^2}\left[\frac{(1-\sigma)^2}{1-2\sigma}\pi^2 + \frac{\sigma(1-\sigma)}{1-2\sigma}\right]\left(\frac{c}{a}\right)^2 \right\}$$

Minizing the strain energy with respect to f, & g,

$$\frac{(1-\sigma)^2}{(1-2\sigma)}\left\{2f+\frac{2}{\pi}\left(g+\frac{1}{n^2}\right)\right\}+\frac{(1-\sigma)}{2}\left\{\frac{2}{\pi}g-\frac{4}{\pi}\left(g+\frac{1}{n^2}\right)\right\}$$

$$+\frac{\left(\frac{t}{a}\right)^2}{12}\left\{\left(\pi+\frac{1-\sigma}{2\pi}\right)+\frac{1-\sigma}{2\pi}g\right\}+\cancel{\left(\frac{c}{a}\right)^2(n\pi)^2}=0$$

$$\frac{(1-\sigma)^2}{1-2\sigma}\left\{\frac{2}{\pi}f+\frac{2}{\pi^2}\left(g+\frac{1}{n^2}\right)\right\}+\frac{(1-\sigma)}{2}\left\{\pi\cdot 2g+\frac{2}{\pi}f-\frac{4}{\pi}f\right\}$$

$$+\frac{\left(\frac{t}{a}\right)^2}{12}\left\{\left[\frac{2}{\pi^2}+\frac{-9}{2}-\frac{3\sigma}{2}\right]+2\left(\frac{1}{\pi^2}+\pi\frac{5(1-\sigma)}{2}\right)g+\frac{1-\sigma}{2\pi}f\right\}$$

$$+\left(\frac{c}{a}\right)^2\frac{(n\pi)^2\left(\frac{t}{a}\right)^2}{12\times 32}\left\{2(1+2\sigma)\left(1+\frac{1}{\pi^2}\right)-\frac{1-\sigma}{\pi^2}(1+2g)\right\}=0.$$

$$\frac{2(1-\sigma)^2}{1-2\sigma}f+\left\{\frac{2(1-\sigma)^2}{(1-2\sigma)}\frac{1}{\pi}+\frac{(1-\sigma)}{\pi}-\frac{2(1-\sigma)}{\pi}+\frac{(1-\sigma)}{24\pi}\left(\frac{t}{a}\right)^2\right\}g$$

$$+\frac{(1-\sigma)^2}{n^2(1-2\sigma)}\frac{2}{\pi}-\frac{2(1-\sigma)}{\pi n^2}+\frac{\left(\frac{t}{a}\right)^2}{12}\left(\pi+\frac{1-\sigma}{2\pi}\right)=0.$$

or $\cancel{\frac{2(1-\sigma)}{1-2\sigma}f+\left\{\frac{2(1-\sigma)}{1-2\sigma}\frac{1}{\pi}-\frac{(1-\sigma)}{\pi}\right.}$

$$\frac{2(1-\sigma)}{1-2\sigma}f+\frac{1}{\pi}\left\{\frac{2(1-\sigma)}{1-2\sigma}-1+\frac{1}{24}\left(\frac{t}{a}\right)^2\right\}g+\left\{\frac{1-\sigma}{1-2\sigma}\frac{2}{\pi}-\frac{2}{n^2\pi}+\frac{\left(\frac{t}{a}\right)^2}{12}\left(\frac{\pi}{1-\sigma}+\frac{1}{2\pi}\right)\right\}=0$$

$$\boxed{\frac{2(1-\sigma)}{1-2\sigma}f+\frac{1}{\pi}\left\{\frac{1}{1-2\sigma}+\frac{1}{24}\left(\frac{t}{a}\right)^2\right\}g+\left\{\frac{2}{\pi}\left(\frac{1-\sigma}{1-2\sigma n^2}-\frac{1}{n^2}\right)+\frac{\left(\frac{t}{a}\right)^2}{12}\left(\frac{\pi}{1-\sigma}+\frac{1}{2\pi}\right)\right\}=0.}$$

$$\left\{\frac{2(1-\sigma)^2}{(1-2\sigma)}\frac{1}{\pi} - \frac{(1-\sigma)}{\pi} + \frac{(1-\sigma)}{24\pi}\left(\frac{t}{a}\right)^2\right\} f + \left\{\frac{2(1-\sigma)^2}{1-2\sigma}\frac{1}{\pi^2} + (1-\sigma) + \frac{\left(\frac{t}{a}\right)^2}{6}\left(\frac{1}{\pi^2}+\frac{5(1-\sigma)}{2}\right)\right.$$

$$\left. - \left(\frac{c}{a}\right)^2 \frac{(n\pi)^2\left(\frac{t}{a}\right)^2}{6\times 32}\ \frac{(1-\sigma)}{\pi}\right\} g + \left\{\frac{2(1-\sigma)^2}{(1-2\sigma)\pi^2 n^2} + \frac{\left(\frac{t}{a}\right)^2}{12}\left(\frac{2}{\pi^2}+\frac{9}{2}-\frac{3\sigma}{2}\right)\right.$$

$$\left. + \left(\frac{c}{a}\right)^2 \frac{(n\pi)^2\left(\frac{t}{a}\right)^2}{6\times 32}\ (1+2\sigma)\left(1+\frac{1}{\pi^2}\right)\right\} = 0.$$

$$\left\{\frac{2(1-\sigma)}{(1-2\sigma)} - 1 + \frac{\left(\frac{t}{a}\right)^2}{24}\right\} f + \left\{\frac{2(1-\sigma)}{\pi(1-2\sigma)} + \pi + \frac{\left(\frac{t}{a}\right)^2\pi}{6}\left\{\frac{1}{\pi^2(1-\sigma)}+\frac{5}{2}\right\} - \frac{(n\pi)^2\left(\frac{t}{a}\right)^2}{6\times 32}\left(\frac{c}{a}\right)^2\right\} g$$

$$+ \left\{\frac{2(1-\sigma)}{(1-2\sigma)\pi n^2} + \frac{\left(\frac{t}{a}\right)^2\pi}{12(1-\sigma)}\left(\frac{2}{\pi^2}+\frac{9}{2}-\frac{3\sigma}{2}\right) + \frac{(n\pi)^2\left(\frac{t}{a}\right)^2\left(\pi+\frac{1}{\pi}\right)}{6\times 32}\left(\frac{1+2\sigma}{1-\sigma}\right)\left(\frac{c}{a}\right)^2\right\} = 0$$

$$\left\{\frac{1}{1-2\sigma} + \frac{\left(\frac{t}{a}\right)^2}{24}\right\} f + \left\{\overbrace{\frac{3-4\sigma}{\pi(1-2\sigma)}}^{\frac{2(1-\sigma)}{\pi(1-2\sigma)}+\pi} + \frac{\left(\frac{t}{a}\right)^2}{6}\left\{\frac{1}{\pi(1-\sigma)}+\frac{5\pi}{2}\right\} - \frac{(n\pi)^2\left(\frac{t}{a}\right)^2\left(\frac{c}{a}\right)^2}{192}\right\} g$$

$$+ \left\{\frac{2(1-\sigma)}{(1-2\sigma)n^2\pi} + \frac{\left(\frac{t}{a}\right)^2\pi}{12(1-\sigma)}\left(\frac{2}{\pi^2}+\frac{9}{2}-\frac{3\sigma}{2}\right) + \frac{(n\pi)^2\left(\frac{t}{a}\right)^2\left(\pi+\frac{1}{\pi}\right)}{6\times 32}\left(\frac{1+2\sigma}{1-\sigma}\right)\left(\frac{c}{a}\right)^2\right\} = 0$$

Approximate relation, $1 \gg \frac{t}{a} \gg$, $\frac{c}{a} \ll 1$, $n < \frac{t}{a}$.

$$\frac{2(1-\sigma)}{1-2\sigma} f + \frac{1}{\pi}\left\{\frac{4}{1-2\sigma}\right\} g + \frac{2}{\pi n^2}\left(\frac{\sigma}{1-2\sigma}\right) = 0$$

or $$2(1-\sigma) f + \frac{1}{\pi} g + \frac{2\sigma}{\pi n^2} = 0.$$

$$f + \frac{3-4\sigma}{\pi} g + \frac{2(1-\sigma)}{\pi n^2} = 0.$$

$$\cancel{2(1-\sigma)f + \frac{1}{\pi}g + \frac{2\sigma}{\pi n^2} = 0}$$

$$\cancel{f + \frac{1}{\pi}\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\}g + \frac{2(1-\sigma)}{\pi n^2} = 0.}$$

$$\boxed{\begin{aligned} &2(1-\sigma)f + \frac{1}{\pi}g + \frac{2\sigma}{\pi n^2} = 0 \\ &f + \frac{1}{\pi}\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\}g + \frac{2(1-\sigma)}{\pi n^2} = 0. \end{aligned}}$$

$$\therefore \quad \left\{2(1-\sigma)\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\} - 1\right\}f = \frac{1}{\pi n^2}\left[2(1-\sigma) - \frac{2\sigma}{\left\{2(1-\sigma)+\pi^2(1-2\sigma)\right\}}\right]$$

$$n^2 f\pi = \frac{2(1-\sigma) - 4\sigma(1-\sigma) - 2\pi^2\sigma(1-2\sigma)}{2(1-\sigma)\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\} - 1}$$

$$\boxed{n^2 f = \frac{1}{\pi}\,\frac{2(1-2\sigma)\left\{(1-\sigma) - \pi^2\sigma\right\}}{2(1-\sigma)\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\} - 1} = l}$$

$$\cancel{\frac{1}{\pi}}$$

$$\frac{1}{\pi}\left\{2(1-\sigma)\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\} - 1\right\}g = \frac{1}{\pi n^2}\left\{2\sigma - 4(1-\sigma)^2\right\}$$

$$\boxed{n^2 g = \frac{2\left\{\sigma - 2(1-\sigma)^2\right\}}{2(1-\sigma)\left\{2(1-\sigma) + \pi^2(1-2\sigma)\right\} - 1} = m}$$

Minimizing with respect to $\left(\frac{C}{a}\right)^2$

$$\frac{(1-\sigma)^2}{(1-2\sigma)}\left\{f^2+\frac{2}{\pi}f\left(g+\frac{1}{n^2}\right)+\frac{1}{\pi^2}\left(g+\frac{1}{n^2}\right)^2\right\}+\frac{(1-\sigma)}{2}\left\{g^2+\frac{2}{\pi}fg-\frac{4}{\pi}f\left(g+\frac{1}{\pi^2}\right)\right\}$$

$$+\frac{(1-\sigma)^2}{(1-2\sigma)}\frac{1}{16}\left(\frac{C}{a}\right)^2\left(18\pi^2+1+\frac{9}{2\pi^2}\right)$$

$$+\frac{\left(\frac{t}{a}\right)^2}{12}\left\{\left[\frac{\sigma}{n^2}+\left(\pi+\frac{1}{\pi}\right)^2\right]+\left(\frac{2}{\pi^2}+\frac{9}{2}-\frac{3\sigma}{2}\right)g+\left(\pi+\frac{1-\sigma}{2\pi}\right)f+\left(\frac{1}{\pi^2}+\frac{5(1-\sigma)}{2}\right)g^2\right.$$

$$\left.+\frac{1-\sigma}{2\pi}fg\right\}+\frac{\left(\frac{C}{a}\right)^2(n\pi)^2(1+g)\left(\frac{t}{a}\right)^2}{192}\left\{2(1+2\sigma)\left(1+\frac{1}{\pi^2}\right)-\frac{(1-\sigma)}{\pi^2}g\right\}$$

$$=\frac{e}{n^2\pi^2}\left[\frac{(1-\sigma)^2}{1-2\sigma}\pi^2+\frac{\sigma(1-\sigma)}{1-2\sigma}\right]=\frac{e}{n^2\pi^2}\frac{(1-\sigma)}{(1-2\sigma)}\left\{\pi^2(1-\sigma)+\sigma\right\}$$

$$\sigma = 0.300,\qquad \frac{1-0.300}{1-0.600}=\frac{0.700}{0.400}=$$

$$\frac{1}{\pi^2}\frac{(1-\sigma)}{(1-2\sigma)}\left\{\pi^2(1-\sigma)+\sigma\right\}=\frac{1}{\pi^2}\cdot\frac{0.700}{0.400}\left\{0.700\,\pi^2+0.300\right\}$$

$$=\frac{0.700}{0.400}\left\{0.700+\frac{0.300}{\pi^2}\right\}=\frac{0.700}{0.400}\times 0.7304 = 1.278$$

$$\frac{2(1-\sigma)}{1-2\sigma} f + \frac{1}{\pi}\left\{\frac{1}{1-2\sigma} + \frac{\left(\frac{t}{a}\right)^2}{24}\right\} g + \left\{\frac{2}{\pi n^2}\left(\frac{\sigma}{1-2\sigma}\right) + \frac{\left(\frac{t}{a}\right)^2}{12}\left(\frac{\pi}{1-\sigma} + \frac{1}{2\pi}\right)\right\} = 0.$$

Using the approximate relation:

$$n^2 f = \frac{1}{\pi}\frac{2\times 0.4\,(0.7 - 0.3\pi^2)}{2\times 0.7\{1.4 + \pi^2\times 0.4\} - 1} = \frac{1}{\pi}\frac{0.4\,(0.7 - 0.3\pi^2)}{0.7(1.4 + \pi^2\times 0.4) - 0.5}$$

$$= \frac{1}{\pi}\frac{-0.4\times 2.256}{3.240} = -\frac{0.2786}{\pi}$$

$$n^2 g = \frac{2\{0.3 - 2\times 0.49\}}{2\times 0.7\{1.4 + \pi^2\times 0.4\} - 1} = -\frac{0.68}{3.240} = -0.2098$$

Neglected the $\left(\frac{t}{a}\right)^2$ terms.

$$\frac{1}{n^4}\frac{0.49}{0.4}\left\{\left(\frac{0.2786}{\pi}\right)^2 - \frac{2\times 0.2786}{\pi^2}(1 - 0.2098) + \frac{1}{\pi^2}(1 - 0.2098)^2\right\}$$

$$+ \frac{1}{n^4}\frac{0.7}{2}\left\{0.2098^2 + \frac{2}{\pi^2}0.2786\times 0.2098 + \frac{4}{\pi^2}0.2786(1 - 0.2098)\right\}$$

$$+ \frac{\left(\frac{t}{a}\right)^2}{12}\left\{\left[\frac{0.3}{n^2} + \left(\pi + \frac{1}{\pi}\right)^2\right] + \left[\frac{2}{\pi^2} + \frac{9}{2} - \frac{3\times 0.3}{2}\right]0.2098 - \left(1 + \frac{0.7}{\pi^2}\right)0.2786\right.$$

$$\left. + \left(\frac{1}{\pi^2} + \frac{5\times 0.7}{2}\right)0.2098^2 + \frac{0.7}{2\pi^2}0.2786\times 0.2098\right\}$$

$$= \frac{1.278}{n^2\pi^2}\,e$$

$$\frac{1}{n^2}\left[\frac{0.49}{0.4}\left\{0.2786^2 - 2\times 0.2786\times 0.7902 + 0.7902^2\right\}\right.$$

$$\left. + 0.35\left\{(0.2098\pi)^2 + 2\times 0.2786\times 0.2098 + 4\times 0.2786\times 0.7902\right\}\right]$$

$$+ n^2\left[\frac{\left(\frac{t}{a}\right)^2}{12}\pi^2\left\{(\pi+\tfrac{1}{\pi})^2 - \left[\tfrac{2}{\pi^2}+\tfrac{9}{2} - \tfrac{0.9}{2}\right]0.2098 - \left(1+\tfrac{0.7}{\pi^2}\right)0.2786\right.\right.$$

$$\left.\left. + \left(\tfrac{1}{\pi^2} + 5\times 0.35\right)\frac{0.2098^2}{\pi^4} + 0.35\,\frac{0.2786\times 0.2098}{\pi^2}\right\}\right] \cancel{=}$$

$$+ 0.025\pi^2\left(\frac{t}{a}\right)^2 = e.$$

$$\frac{1}{n^2}\left[0.4274\times\frac{0.49}{0.40} + 0.35\times 1.429\right]$$

$$+ n^2\left(\frac{t}{a}\right)^2\times 0.821\left\{11.96 - 0.806 - 0.298 + 0.0814 + 0.00207\right\}$$

$$+ 0.025\pi^2\left(\frac{t}{a}\right)^2 = e\cdot 1.278$$

$$\frac{1.024}{n^2} + n^2\left(\frac{t}{a}\right)^2\times 8.99 + 0.2464\left(\frac{t}{a}\right)^2 = e\ 1.278$$

$$-\frac{1.024}{n^4} + 8.99\left(\frac{t}{a}\right)^2 = 0.$$

$$\frac{1}{n^4} = \frac{8.99\left(\frac{t}{a}\right)^2}{1.024}$$

$$\frac{1}{n^2} = 2.96\left(\frac{t}{a}\right)$$

$$\boxed{\sigma_{cr} = 0.529E\left(\frac{t}{a}\right) + 0.0216E\left(\frac{t}{a}\right)^2}$$

$$\pi^2\,1.278\,e = \cancel{5.92}\ 6.07\left(\frac{t}{a}\right) + 0.2464\left(\frac{t}{a}\right)^2$$

$$e = \cancel{0.467}\ 0.481\left(\frac{t}{a}\right) + 0.0196\left(\frac{t}{a}\right)^2$$

If we take the buckling wave form as

~~$u = C\alpha\{C_1$~~

$$u = C\left\{\alpha_1 \sin n\theta \cos\frac{n\pi x}{a} + \alpha_3 \sin 3n\theta \cos\frac{3n\pi x}{a}\right\}$$

$$v = C\left\{\beta_1 \cos n\theta \sin\frac{n\pi x}{a} + \beta_3 \cos 3n\theta \sin\frac{3n\pi x}{a}\right\}$$

$$w = C\left\{\sin n\theta \sin\frac{n\pi x}{a} + \gamma_3 \sin 3n\theta \sin\frac{3n\pi x}{a}\right\}$$

$$\epsilon_1 + \epsilon_2 = \frac{\partial u}{\partial x} + \frac{1}{a}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right\}$$

$$= C\Big[-\alpha_1\left(\frac{n\pi}{a}\right)\sin n\theta \sin\frac{n\pi x}{a} - \alpha_3\left(\frac{3n\pi}{a}\right)\sin 3n\theta \sin\frac{3n\pi}{a}$$

$$-\frac{1}{a}(n\beta_1 + 1)\sin n\theta \sin\frac{n\pi x}{a} - \frac{1}{a}(3n\beta_3 + 1)\sin 3n\theta \sin\frac{3\pi x}{a}\Big]$$

$$+ \frac{C^2}{2}\Big[\left\{\left(\frac{n\pi}{a}\right)\sin n\theta \cos\frac{n\pi x}{a} + \gamma_3\left(\frac{3n\pi}{a}\right)\sin 3n\theta \cos\frac{3n\pi x}{a}\right\}^2 + \frac{1}{a^2}\Big\{n \cos n\theta \sin\frac{n\pi x}{a}$$

$$+ \gamma_3 3n \cos 3n\theta \sin\frac{3n\pi x}{a}\Big\}^2\Big]$$

$$= \left(\frac{C}{a}\right)\Big[-(n\pi\alpha_1 + n\beta_1 + 1)\sin n\theta \sin\frac{n\pi x}{a} - (3n\pi\alpha_3 + 3n\beta_3 + 1)\sin 3n\theta \sin\frac{3n\pi x}{a}\Big]$$

$$+ \frac{1}{2}\left(\frac{C}{a}\right)^2\Big[(n\pi)^2\left\{\sin n\theta \cos\frac{n\pi x}{a} + 3\sin 3n\theta \cos\frac{3n\pi x}{a}\right\}^2$$

$$+ n^2\left\{\cos n\theta \sin\frac{n\pi x}{a} + 3\cos 3n\theta \sin\frac{3n\pi x}{a}\right\}^2\Big]$$

$$= -\left(\frac{Cn}{a}\right)\Big[\left(\pi\alpha_1 + \beta_1 + \frac{1}{n}\right)\sin n\theta \sin\frac{n\pi x}{a} + \left(3\pi\alpha_3 + 3\beta_3 + \frac{1}{n}\right)\sin 3n\theta \sin\frac{3n\pi x}{a}\Big]$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2\Big[\pi^2\left\{\sin n\theta \cos\frac{n\pi x}{a} + 3\gamma_3 \sin 3n\theta \cos\frac{3n\pi x}{a}\right\}^2 + \left\{\cos n\theta \sin\frac{n\pi x}{a} + 3\gamma_3 \cos 3n\theta \sin\frac{3n\pi x}{a}\right\}^2\Big]$$

$$A^2 - 4\epsilon_1\epsilon_2 = \left\{ \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta} + \frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} \right\}^2 - 4\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\left[\frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a} + \frac{1}{2a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]$$

$$= \left[\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right]^2 + \frac{2}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\left(\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right) - \frac{4}{a}\left[\frac{\partial u}{\partial x}\right]\left[\frac{\partial v}{\partial \theta} - w\right]$$

$$- \frac{2}{a}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial v}{\partial \theta} - w\right) - \frac{2}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\frac{\partial u}{\partial x}$$

The effective terms will be

$$\left(\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right)^2 - \frac{4}{a}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial \theta} - w\right)$$

If we take the buckling wave form as

$$u = C\left\{\alpha_1 \sin n\theta \cos\frac{n\pi x}{a} + \alpha_2 \cos 2n\theta \sin\frac{2n\pi x}{a}\right\}$$

$$v = C\left\{\beta_1 \cos n\theta \sin\frac{n\pi x}{a} + \beta_2 \sin 2n\theta \cos\frac{2n\pi x}{a}\right\}$$

$$w = C\left\{\sin n\theta \sin\frac{n\pi x}{a} + \gamma_2 \cos 2n\theta \cos\frac{2n\pi x}{a}\right\}$$

$$\epsilon_1 + \epsilon_2 = \frac{\partial u}{\partial x} + \frac{1}{a}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right\}$$

$$= C\left\{-\left(\frac{n\pi}{a}\right)\alpha_1 \sin n\theta \sin\frac{n\pi x}{a} + \alpha_2\left(\frac{2n\pi}{a}\right)\cos 2n\theta \cos\frac{2n\pi x}{a}\right\}$$

$$+ C\left\{-\frac{1}{a}(n\beta_1 + 1)\sin n\theta \sin\frac{n\pi x}{a} + \frac{1}{a}(2n\beta_2 - \gamma_2)\cos 2n\theta \cos\frac{2n\pi x}{a}\right\}$$

$$+ \frac{1}{2}\left[C^2\left(\frac{n\pi}{a}\right)^2\left\{+\sin n\theta \sin\frac{n\pi x}{a} - 2\gamma_2 \cos 2n\theta \sin\frac{2n\pi x}{a}\right\}^2\right.$$

$$\left. + C^2\frac{1}{a^2}n^2\left\{\cos n\theta \sin\frac{n\pi x}{a} - 2\gamma_2 \sin 2n\theta \cos\frac{2n\pi x}{a}\right\}^2\right]$$

$$= -\left(\frac{Cn}{a}\right)\left[\left(\pi\alpha_1 + \beta_1 + \frac{1}{n}\right)\sin n\theta \sin\frac{n\pi x}{a} - \left(2\pi\alpha_2 + 2\beta_2 - \frac{\gamma_2}{n}\right)\cos 2n\theta \cos\frac{2n\pi x}{a}\right]$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2\left[\pi^2\left\{\sin n\theta \cos\frac{n\pi x}{a} - 2\gamma_2 \cos 2n\theta \sin\frac{2n\pi x}{a}\right\}^2\right.$$

$$\left. + \left\{\cos n\theta \sin\frac{n\pi x}{a} - 2\gamma_2 \sin 2n\theta \cos\frac{2n\pi x}{a}\right\}^2\right]$$

$$\epsilon_1+\epsilon_2 = -\left(\frac{Cn}{a}\right)\left[\left(\pi\alpha_1+\beta_1+\frac{1}{n}\right)\sin n\theta \sin\frac{n\pi x}{a} - \left(2\pi\alpha_2+2\beta_2-\frac{\gamma_2}{n}\right)\cos 2n\theta\cos\frac{2n\pi x}{a}\right] \tag{42}$$

$$+\frac{1}{2}\left(\frac{Cn}{a}\right)^2\Big[(\pi^2+1)\left(\gamma_2^2+\frac{1}{4}\right)-\frac{(\pi^2-1)}{4}\left(\cos 2n\theta-\cos\frac{2n\pi x}{a}\right)-\frac{(\pi^2+1)}{4}\cos 2n\theta\cos\frac{2n\pi x}{a}$$

$$-\gamma_2(\pi^2+1)\sin 3n\theta\sin\frac{3n\pi x}{a}+\gamma_2(\pi^2-1)\sin n\theta\sin\frac{3n\pi x}{a}$$

$$-\gamma_2(\pi^2-1)\sin 3n\theta\sin\frac{n\pi x}{a}+\gamma_2(\pi^2+1)\sin n\theta\sin\frac{n\pi x}{a}$$

$$+(\pi^2-1)\gamma_2^2\left(\cos 4n\theta-\cos\frac{4n\pi x}{a}\right)+\gamma_2^2(\pi^2+1)\cos 4n\theta\cos\frac{4n\pi x}{a}\Big]$$

$$=-\left(\frac{Cn}{a}\right)\Big[\left\{\pi\alpha_1+\beta_1+\frac{1}{n}-\frac{1}{2}\left(\frac{Cn}{a}\right)\gamma_2(\pi^2+1)\right\}\sin n\theta\sin\frac{n\pi x}{a}$$

$$-\left\{2\pi\alpha_2+2\beta_2-\frac{\gamma_2}{n}-\frac{1}{2}\left(\frac{Cn}{a}\right)\frac{\pi^2+1}{4}\right\}\cos 2n\theta\cos\frac{2n\pi x}{a}\Big]$$

$$+\frac{1}{2}\left(\frac{Cn}{a}\right)^2\Big[(\pi^2+1)\left(\gamma_2^2+\frac{1}{4}\right)-\frac{(\pi^2-1)}{4}\left(\cos 2n\theta-\cos\frac{2n\pi x}{a}\right)$$

$$-\gamma_2(\pi^2+1)\sin 3n\theta\sin\frac{3n\pi x}{a}+\gamma_2(\pi^2-1)\sin n\theta\sin\frac{3n\pi x}{a}-\gamma_2(\pi^2-1)\sin 3n\theta\sin\frac{n\pi x}{a}$$

$$+(\pi^2-1)\gamma_2^2\left(\cos 4n\theta-\cos\frac{4n\pi x}{a}\right)+\gamma_2^2(\pi^2+1)\cos 4n\theta\cos\frac{4n\pi x}{a}\Big]$$

$$\iint(\epsilon_1+\epsilon_2)^2\,dx\,d\theta$$

$$=\left(\frac{Cn}{a}\right)^2\Big[\left\{\pi\alpha_1+\beta_1+\frac{1}{n}-\frac{\gamma_2}{2}\left(\frac{Cn}{a}\right)(\pi^2+1)\right\}^2\pi\cdot\frac{a}{n}+\left\{2\pi\alpha_2+2\beta_2-\frac{\gamma_2}{n}-\frac{1}{8}\left(\frac{Cn}{a}\right)(\pi^2+1)\right\}^2\pi\cdot\frac{a}{n}\Big]$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^4\Big[(\pi^2+1)^2\left(\gamma_2^2+\frac{1}{4}\right)^2 2\pi\cdot\frac{2a}{n}+\frac{(\pi^2-1)^2}{16}\left(\pi\cdot\frac{2a}{n}+2\pi\cdot\frac{a}{n}\right)$$

$$+\gamma_2^2(\pi^2+1)^2\pi\cdot\frac{a}{n}+\gamma_2^2(\pi^2-1)^2\left\{\pi\cdot\frac{a}{n}+\pi\cdot\frac{a}{n}\right\}+\gamma_2^4(\pi^2-1)^2 4\pi\cdot\frac{a}{n}+\gamma_2^4(\pi^2+1)^2\pi\cdot\frac{a}{n}\Big]$$

$$\iint (\epsilon_1+\epsilon_2)^2\,dx\,d\theta$$

$$= \left(\frac{Cn}{a}\right)^2\frac{\pi a}{n}\Big[\Big\{\pi\alpha_1+\beta_1+\frac{1}{n}-\frac{\gamma_2}{2}\left(\frac{Cn}{a}\right)(\pi^2+1)\Big\}^2+\Big\{2\pi\alpha_2+2\beta_2-\frac{\gamma_2}{n}-\frac{1}{8}\left(\frac{Cn}{a}\right)(\pi^2+1)\Big\}^2$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\Big\{(\pi^2+1)^2\Big[4\left(\gamma_2^2+\frac{1}{4}\right)^2+\gamma_2^2+\gamma_2^4\Big]+(\pi^2-1)^2\Big[\frac{1}{4}+2\gamma_2^2+4\gamma_2^4\Big]\Big\}\Big]$$

$$= \left(\frac{Cn}{a}\right)^2\frac{\pi a}{n}\Big[\Big\{\pi\alpha_1+\beta_1+\frac{1}{n}-\frac{\gamma_2}{2}\left(\frac{Cn}{a}\right)(\pi^2+1)\Big\}^2+\Big\{2\pi\alpha_2+2\beta_2-\frac{\gamma_2}{n}-\frac{1}{8}\left(\frac{Cn}{a}\right)(\pi^2+1)\Big\}^2$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\Big\{8(\pi^4+1)\left(\gamma_2^2+\frac{1}{4}\right)^2+(\pi^2+1)^2\gamma_2^2(1+\gamma_2^2)\Big\}\Big]$$

$$\epsilon^2-4\epsilon_1\epsilon_2=\Big[\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial \theta}\Big]^2-\frac{4}{a}\frac{\partial u}{\partial x}\Big(\frac{\partial v}{\partial \theta}-w\Big)+\frac{2}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\Big(\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial \theta}\Big)$$

$$-\frac{2}{a}\Big(\frac{\partial w}{\partial x}\Big)^2\Big(\frac{\partial v}{\partial \theta}-w\Big)-\frac{2}{a^2}\frac{\partial u}{\partial x}\Big(\frac{\partial w}{\partial \theta}\Big)^2$$

$$= C^2\left(\frac{n}{a}\right)^2\Big[\pi\beta_1\cos n\theta\cos\frac{n\pi x}{a}-2\pi\beta_2\sin 2n\theta\sin\frac{2n\pi x}{a}$$

$$+\alpha_1\cos n\theta\cos\frac{n\pi x}{a}-2\alpha_2\sin 2n\theta\sin\frac{2n\pi x}{a}\Big]^2$$

$$-C^2\left(\frac{n}{a}\right)^2\Big[-\pi\alpha_1\sin n\theta\sin\frac{n\pi x}{a}+2\pi\alpha_2\cos 2n\theta\cos\frac{2n\pi x}{a}\Big]$$

$$\times(-)\Big[\Big(\pi\beta_1+\frac{1}{n}\Big)\sin n\theta\sin\frac{n\pi x}{a}-\Big(2\beta_2-\frac{\gamma_2}{n}\Big)\cos 2n\theta\cos\frac{2n\pi x}{a}\Big]$$

$$+2\left(\frac{Cn}{a}\right)^3\Big[\pi\sin n\theta\cos\frac{n\pi x}{a}-2\pi\gamma_2\cos 2n\theta\sin\frac{2n\pi x}{a}\Big]\Big[\cos n\theta\sin\frac{n\pi x}{a}-2\gamma_2\sin 2n\theta\cos\frac{2n\pi x}{a}\Big]$$

$$\Big[(\pi\beta_1+\alpha_1)\cos n\theta\cos\frac{n\pi x}{a}-(2\pi\beta_2+2\alpha_2)\sin 2n\theta\sin\frac{2n\pi x}{a}\Big]$$

$$+2\left(\frac{Cn}{a}\right)^3\Big[\pi\sin n\theta\cos\frac{n\pi x}{a}-2\pi\gamma_2\cos 2n\theta\sin\frac{2n\pi x}{a}\Big]^2\Big[\Big(\beta_1+\frac{1}{n}\Big)\sin n\theta\sin\frac{n\pi x}{a}-\Big(2\beta_2-\frac{\gamma_2}{n}\Big)\cos 2n\theta\cos\frac{2n\pi x}{a}\Big]$$

$$- 2\left(\frac{Cn}{a}\right)^3\left[-\pi\alpha_1 \sin n\theta \sin\frac{n\pi x}{a} + 2\pi\alpha_2 \cos 2n\theta \cos\frac{2n\pi x}{a}\right]\left[\cos n\theta \sin\frac{n\pi x}{a} - 2\gamma_2 \sin 2n\theta \cos\frac{2n\pi x}{a}\right]^2$$

$$= \left(\frac{Cn}{a}\right)^2\left[(\pi\beta_1+\alpha_1)\cos n\theta \cos\frac{n\pi x}{a} - (2\pi\beta_2+2\alpha_2)\sin 2n\theta \sin\frac{2n\pi x}{a}\right]^2$$

$$- \left(\frac{Cn}{a}\right)^2\left[\pi\alpha_1 \sin n\theta \sin\frac{n\pi x}{a} - 2\pi\alpha_2 \cos 2n\theta \cos\frac{2n\pi x}{a}\right]\left[\left(\beta_1+\frac{1}{n}\right)\sin n\theta \sin\frac{n\pi x}{a} - \left(2\beta_2-\frac{\gamma_2}{n}\right)\cos 2n\theta \cos\frac{2n\pi x}{a}\right]$$

$$+ 2\left(\frac{Cn}{a}\right)^3\left[\frac{\pi}{4}\sin 2n\theta \sin\frac{2n\pi x}{a} - \pi\gamma_2 \cos n\theta \cos\frac{n\pi x}{a} + \pi\gamma_2 \cos 3n\theta \cos\frac{3n\pi x}{a}\right.$$
$$\left. + \pi\gamma_2^2 \sin 4n\theta \sin\frac{4n\pi x}{a}\right]\left[(\pi\beta_1+\alpha_1)\cos n\theta \cos\frac{n\pi x}{a} - (2\pi\beta_2+2\alpha_2)\sin 2n\theta \sin\frac{2n\pi x}{a}\right]$$

$$+ 2\left(\frac{Cn}{a}\right)^3\left[\left(\beta_1+\frac{1}{n}\right)\sin n\theta \sin\frac{n\pi x}{a} - \left(2\beta_2-\frac{\gamma_2}{n}\right)\cos 2n\theta \cos\frac{2n\pi x}{a}\right]$$

$$\pi^2\left[\frac{1}{4} - \frac{1}{4}\left(\cos 2n\theta - \cos\frac{2n\pi x}{a}\right) - \underline{\frac{1}{4}\cos 2n\theta \cos\frac{2n\pi x}{a}}\right.$$
$$\left. + \underline{\gamma_2 \sin n\theta \sin\frac{n\pi x}{a}} + \cdots\right]$$

$$+\cdots$$

$$+ 2\left(\frac{Cn}{a}\right)^3\left[-\frac{1}{4}\cos 2n\theta \cos\frac{2n\pi x}{a} + \gamma_2 \sin n\theta \sin\frac{n\pi x}{a}\right]\left[\pi\alpha_1 \sin n\theta \sin\frac{n\pi x}{a} - 2\pi\alpha_2 \cos 2n\theta \cos\frac{2n\pi x}{a}\right]$$

$$\# \quad \iint (\tau^2 - 4t_1 t_2)\, dx\, d\theta$$

$$= \left(\frac{Cn}{a}\right)^2\left[(\pi\beta_1+\alpha_1)^2 \pi\frac{a}{n} + (2\pi\beta_2+2\alpha_2)^2 \pi\frac{a}{n}\right]$$

$$- \left(\frac{Cn}{a}\right)^2\left[\pi\alpha_1\left(\beta_1+\frac{1}{n}\right)\pi\cdot\frac{a}{n} + 2\pi\alpha_2\left(2\beta_2-\frac{\gamma_2}{n}\right)\pi\cdot\frac{a}{n}\right]$$

$$+ 2\left(\frac{Cn}{a}\right)^3\left[-\frac{\pi}{4}(2\pi\beta_2+2\alpha_2)\pi\cdot\frac{a}{n} - \pi\gamma_2(\pi\beta_1+\alpha_1)\pi\cdot\frac{a}{n}\right]$$

$$+ 2\left(\frac{Cn}{a}\right)^3\pi^2\left[\frac{1}{4}\left(2\beta_2-\frac{\gamma_2}{n}\right)\pi\cdot\frac{a}{n} + \gamma_2\left(\beta_1+\frac{1}{n}\right)\pi\cdot\frac{a}{n}\right]$$

$$+2\left(\frac{Cn}{a}\right)^3\pi\left[\cancel{\frac{1}{4}\, 2\alpha_2\, \pi\,\frac{a}{n}} + \alpha_1\gamma_2\, \pi\cdot\frac{a}{n}\right]$$

$$(e^2 - 4e_1 e_2)\, dx\, d\theta$$

$$= \left(\frac{\pi a}{n}\right)\left(\frac{Cn}{a}\right)^2 \left[\pi^2\beta_1^2 + \pi\alpha_1\beta_1 + \alpha_1^2 + \frac{\pi\alpha_1}{n} + 4\pi^2\beta_2^2 + 4\pi\alpha_2\beta_2 + 4\alpha_2^2 + \frac{2\pi\alpha_2\gamma_2}{n} \right]$$

$$+ 2\left(\frac{Cn}{a}\right)^3 \left[\pi^2\left\{\left(\frac{\beta_2 - \frac{\beta_2}{n}}{4}\right) + \gamma_2\left(\cancel{\beta_1} + \frac{1}{n}\right)\right\} - \cancel{\frac{\pi}{4}(2\pi\beta_2)} - \pi\gamma_2\cancel{(\pi\beta_1)} \right] \frac{\pi a}{n}$$

$$= \left(\frac{\pi a}{n}\right)\left(\frac{Cn}{a}\right)^2 \pi^2 \Big[\beta_1^2 + \frac{\alpha_1\beta_1}{\pi} + \frac{\alpha_1^2}{\pi^2} + \frac{\alpha_1}{\pi n} + 4\beta_2^2 + \frac{4\alpha_2\beta_2}{\pi} + \frac{4\alpha_2^2}{\pi^2} + \frac{2\alpha_2\gamma_2}{\pi n}$$

$$+ 2\left(\frac{Cn}{a}\right)\left\{\frac{3}{4}\frac{\gamma_2}{n}\right\} \Big]$$

$$(1-\sigma^2)\frac{a^2 W}{Eat} = \left(\frac{\pi a}{n}\right)\left(\frac{Cn}{a}\right)^2 \pi^2 \Big[\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi} - \frac{\gamma_2}{2}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\}^2 + \left\{2\alpha_2 + \frac{2\beta_2}{\pi} - \frac{\gamma_2}{n\pi} - \frac{1}{8}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\}^2$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2 \left\{ 8\left(\pi^2 + \frac{1}{\pi^2}\right)\left(\gamma_2^2 + \frac{1}{4}\right)^2 + \left(\pi + \frac{1}{\pi}\right)^2 \gamma_2^2 (1 + \gamma_2^2) \right\}$$

$$+ \frac{1-\sigma}{2}\left\{ \beta_1^2 + \frac{\alpha_1\beta_1}{\pi} + \frac{\alpha_1^2}{\pi^2} + \frac{\alpha_1}{\pi n} + 4\beta_2^2 + \frac{4\alpha_2\beta_2}{\pi} + \frac{4\alpha_2^2}{\pi^2} + \frac{2\alpha_2\gamma_2}{\pi n} + 2\left(\frac{Cn}{a}\right)\frac{3}{4}\frac{\gamma_2}{n} \right\} \Big]$$

$\frac{\partial W}{\partial \alpha_1} = 0$ gives

$$\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi} - \frac{1}{2}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 + \frac{1-\sigma}{4}\left(\frac{\beta_1}{\pi} + \frac{2\alpha_1}{\pi^2} + \frac{1}{\pi n}\right) = 0$$

$\frac{\partial W}{\partial \beta_1} = 0$ gives

$$\frac{1}{\pi}\left[\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi} - \frac{1}{2}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2\right] + \frac{1-\sigma}{4}\left[2\beta_1 + \frac{\alpha_1}{\pi}\right] = 0$$

$\frac{\partial W}{\partial \alpha_2} = 0$ gives

$$2\left\{2\alpha_2 + \frac{2\beta_2}{\pi} - \frac{\gamma_2}{n\pi} - \frac{1}{8}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\} + \frac{1-\sigma}{4}\left[-\frac{4\beta_2}{\pi} + \frac{8\alpha_2}{\pi^2} - \frac{2\gamma_2}{\pi n}\right] = 0$$

$\frac{\partial W}{\partial \beta_2} = 0$ gives

$$\frac{2}{\pi}\left\{2\alpha_2 + \frac{2\beta_2}{\pi} - \frac{\gamma_2}{n\pi} - \frac{1}{8}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\} + \frac{1-\sigma}{4}\left[8\beta_2 + \frac{4\alpha_2}{\pi}\right] = 0.$$

$\frac{\partial W}{\partial \gamma_2} = 0$ gives.

$$-\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi} - \frac{\gamma_2}{2}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\} - \cancel{\frac{1}{4}\left(\frac{Cn}{a}\right)} - \frac{2}{n\pi}\left\{2\alpha_2 + \frac{2\beta_2}{\pi} - \frac{\gamma_2}{n\pi} - \frac{1}{8}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\}$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{16\left(\pi^2 + \frac{1}{\pi^2}\right)\left(\gamma_2^2 + \frac{1}{4}\right)2\gamma_2 + \left(\pi + \frac{1}{\pi}\right)^2\left(2\gamma_2 + 4\gamma_2^3\right)\right\}$$

$$+ \frac{1-\sigma}{2}\left\{-\frac{2\alpha_2}{\pi n} + 2\left(\frac{Cn}{a}\right)\frac{3}{4n}\right\} = 0$$

or

$$-\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right\} - \frac{2}{n\pi}\left\{2\alpha_2 + \frac{2\beta_2}{\pi} - \frac{\gamma_2}{n\pi} - \frac{1}{8}\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\right\}$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{16\left(\pi^2 + \frac{1}{\pi^2}\right)\left(2\gamma_2^3 + \frac{1}{2}\gamma_2\right) + \left(\pi + \frac{1}{\pi}\right)^2\left(3\gamma_2 + 4\gamma_2^3\right)\right\}$$

$$+ \frac{1-\sigma}{2}\left\{\frac{3}{2n}\left(\frac{Cn}{a}\right) - \frac{2\alpha_2}{n\pi}\right\} = 0.$$

$$\varepsilon = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{\partial u}{\partial x} \qquad 47)$$

$$\left(\frac{\partial w}{\partial x}\right)^4 = \left(\frac{Cn}{a}\right)^2 \pi^2 \Big\{ \left(\tfrac{1}{4}+\gamma_2^2\right) - \frac{1}{4}\left(\cos 2n\theta - \cos\frac{2n\pi x}{a}\right) - \Big(\sin 3n\theta \sin\frac{3n\pi x}{a}$$

$$- \sin n\theta \sin\frac{3n\pi x}{a} + \sin 3n\theta \sin\frac{n\pi x}{a} - \sin n\theta \sin\frac{n\pi x}{a}\Big)$$

$$+ \left(\cos 4n\theta - \cos\frac{4n\pi x}{a}\right) - \cos 4n\theta \cos\frac{4n\pi x}{a} \Big\}$$

$\cancel{\iint \left(\frac{\partial w}{\partial x}\right)^4 d\theta}$ $\quad \iint \left(\frac{\partial w}{\partial x}\right)^4 d\theta\, dx$

$$= \left(\frac{Cn}{a}\right)^4 \pi^4 \Big\{ \left(\tfrac{1}{4}+\gamma_2^2\right)^2 2\pi \frac{2a}{n} + \frac{1}{16}\left(\pi\cdot\frac{2a}{n} + 2\pi\frac{a}{n}\right) + \gamma_2\left(\pi\cdot\frac{a}{n} + \pi\frac{a}{n} + \pi\frac{a}{n} + \pi\frac{a}{n}\right)$$

$$+ \gamma_2^2\left(\pi\cdot\frac{2a}{n} + 2\pi\frac{a}{n}\right) + \gamma_2^2 \pi\cdot\frac{a}{n} \Big\}$$

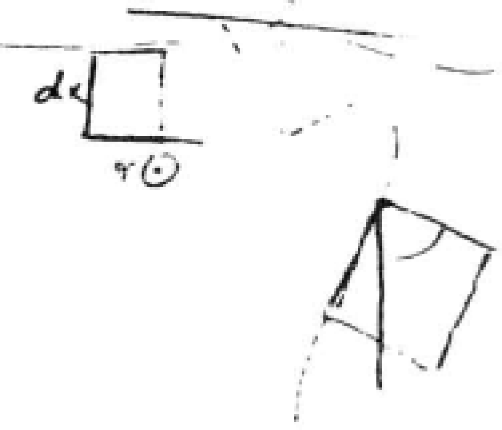
$$= \left(\frac{\pi a}{n}\right)\left(\frac{Cn}{a}\right)^4 \pi^4 \Big\{ 4\left(\frac{1}{4}+\gamma_2^2\right)^2 + \frac{1}{4} + 4\gamma_2 + 4\gamma_2^2 + \gamma_2^2 \Big\}$$

$x = x + u$

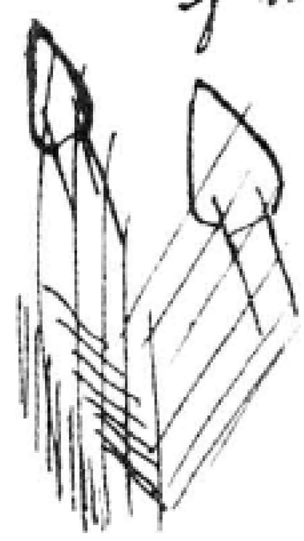
$r' = r + w$

$\theta' = \theta + \frac{v}{r}$

$$\int dx \quad \int_0^{\ell} (1+\varepsilon_x)(\cos\theta)\, dx$$



If we take buckled state to be represented as



$$u = u_0 x + C\left\{\alpha_1 \sin n\theta \cos \frac{n\pi x}{a} + \alpha_2 \cos 2n\theta \sin \frac{2n\pi x}{a}\right\}$$

$$v = C\left\{\beta_1 \cos n\theta \sin \frac{n\pi x}{a} + \beta_2 \sin 2n\theta \cos \frac{2n\pi x}{a}\right\}$$

$$w = w_0 + C\left\{\sin n\theta \sin \frac{n\pi x}{a} + \gamma_2 \cos 2n\theta \cos \frac{2n\pi x}{a}\right\}$$

The <u>additional</u> terms in

$$\iint (\epsilon_1+\epsilon_2)^2 dx\, d\theta \text{ is } \left(u_0 - \frac{w_0}{a}\right)\iint\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{a^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right\}$$

$$= \left(u_0 - \frac{w_0}{a}\right)\left[\left(\frac{Cn}{a}\right)^2 \pi^2 \left\{\pi\cdot\frac{a}{n} + 4\gamma_2^2 \pi\cdot\frac{a}{n}\right\} + \left(\frac{Cn}{a}\right)^2\left\{\pi\frac{a}{n} + 4\gamma_2^2 \pi \frac{a}{n}\right\}\right]$$

$$= \underline{\underline{\left(u_0 - \frac{w_0}{a}\right)\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left(1+4\gamma_2^2\right)\left(\pi^2+1\right)}}$$

to be multiplied by 1

The <u>additional</u> terms in $\iint (\varpi^2 - 4\epsilon_1\epsilon_2)\, dx\, d\theta$ is

$$\frac{4}{a}\cdot \frac{w_0}{\cancel{a}} u_0 \iint dx\, d\theta + \frac{2w_0}{a}\iint\left(\frac{\partial w}{\partial x}\right)^2 dx\, d\theta - \frac{2u_0}{a^2}\iint\left(\frac{\partial w}{\partial \theta}\right)^2 dx\, d\theta$$

$$= \frac{4}{a}\cdot w_0 \cdot u_0 \frac{4\pi a}{n} + \frac{2w_0}{a}\left(\frac{Cn}{\cancel{a}}\right)^2 \pi^2 \left(\frac{\pi a}{n}\right)\left(1+4\gamma_2^2\right) - \frac{2u_0}{\cancel{a}}\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left(1+4\gamma_2^2\right)$$

$$= \cancel{16}\left(\frac{\pi a}{n}\right)\left[16 u_0 \left(\frac{w_0}{a}\right) + 2\left(\frac{Cn}{a}\right)^2\left\{\pi^2 \frac{w_0}{a} - u_0\right\}\left(1+4\gamma^2\right)\right]$$

to be multiplied by $\cancel{p}\ \frac{(1-\sigma)}{2}$

Corrected total energy

$$(1-\sigma^2)\frac{a^2W}{Eat} = \left(\frac{\pi a}{n}\right)\left(\frac{Cn}{a}\right)^2\pi^2\Big[\left\{\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right\}^2 + 4\left\{\alpha_2+\frac{\beta_2}{\pi}-\frac{\gamma_2}{2n\pi}\right\}^2$$

$$-\left(\frac{Cn}{a}\right)\left\{\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{3}{4n\pi}\right)\gamma_2+\frac{1}{2}\left(\alpha_2+\frac{\beta_2}{\pi}\right)\right\}\left(\pi+\frac{1}{\pi}\right)$$

$$+\left(u_0-\frac{w_0}{a}\right)(1+4\gamma_2^2)\left(1+\frac{1}{\pi^2}\right)+\frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi^2+\frac{1}{\pi^2}\right)\left(9\gamma_2^4+\frac{11}{2}\gamma_2^2+\frac{17}{32}\right)\right.$$

$$\left.+\left(2\gamma_2^4+3\gamma_2^2+\frac{1}{16}\right)\right\}$$

$$+\frac{1-\sigma}{2}\left\{\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}+\frac{1}{n}\right)+4\beta_2^2+\frac{\alpha_2}{\pi}\left(4\beta_2+\frac{4\alpha_2}{\pi}-\frac{2\gamma_2}{n}\right)+\frac{3}{2}\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n}\right.$$

$$\left.+2(1+4\gamma_2^2)\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\right\}\Big] + \underline{\underline{\frac{\pi a}{n}\,u_0\left(\frac{w_0}{a}\right)8(1-\sigma)}}$$

$\alpha_2 = \beta_2 = 0$

$$(1-\sigma^2)\frac{a^2W}{Eat}=\frac{\pi a}{n}\left(\frac{Cn}{a}\right)^2\pi^2\Big[\left\{\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right\}^2+\frac{\gamma_2^2}{n^2\pi^2}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{3}{4n\pi}\right)\gamma_2$$

$$+\left(u_0-\frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)(1+4\gamma_2^2)+\frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi^2+\frac{1}{\pi^2}\right)\left(9\gamma_2^4+\frac{11}{2}\gamma_2^2+\frac{17}{32}\right)\right.$$

$$\left.+\left(2\gamma_2^4+3\gamma_2^2+\frac{1}{16}\right)\right\}$$

$$+\frac{1-\sigma}{2}\left\{\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}+\frac{1}{n}\right)+\frac{3}{2}\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n}+2(1+4\gamma_2^2)\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\right\}\Big]$$

$$+\frac{\pi a}{n}\,u_0\,\frac{w_0}{a}\,8(1-\sigma)$$

$$\cancel{(1-\sigma^2)\frac{2W}{Ea^3} = \left(\frac{t}{a}\right)\frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2}$$

$$(1-\sigma^2)\frac{2W_e}{Ea^3}\frac{1}{\left(\frac{t}{a}\right)} = \frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2\Big[\left\{\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right\}^2+\frac{\gamma_2^2}{n^2\pi^2}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{3}{4n\pi}\right)\gamma_2$$

$$+\left(u_0-\frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right)+\frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi^2+\frac{1}{\pi^2}\right)\left(9\gamma_2^4+\frac{17}{2}\gamma_2^2+\frac{17}{32}\right)+\left(2\gamma_2^4+3\gamma_2^2+\frac{1}{16}\right)\right\}$$

$$+\frac{1-\sigma}{2}\left\{\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}-\frac{1}{n}\right)+\frac{3}{2}\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n}+2\left(1+4\gamma_2^2\right)\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\right\}\Big]$$

$$+\,8(1-\sigma)\frac{\pi}{n}\frac{u_0 w_0}{a}$$

Bending + Torsion:

$$2W_b = Da\Big[\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left\{1+16\gamma_2^2\right\}\frac{\pi^4 n^2}{a^2}+\left(\frac{Cn}{a}\right)^2\frac{\pi a}{n}\left\{1+16\gamma_2^2\right\}\frac{n^2}{a^2}$$

$$+\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left(\frac{\beta_1}{a}\right)^2+2\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left\{1+16\gamma_2^2\right\}\left(\frac{n\pi}{a}\right)^2$$

$$+2\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left(\frac{n\pi}{a}\right)^2\frac{\beta_1}{n}+2\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)n\frac{\beta_1}{a^2}$$

$$+(1-\sigma)\Big\{2\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left(\frac{n\pi}{a}\right)^2\left(1+\frac{\beta_1}{\pi n}\right)^2+32\left(\frac{Cn}{a}\right)^2\gamma_2^2\left(\frac{\pi a}{n}\right)\left(\frac{n\pi}{a}\right)^2$$

$$-2\left(\frac{Cn}{a}\right)^2\left(\frac{\pi a}{n}\right)\left(\frac{n\pi}{a}\right)^2\left(1+\frac{\beta_1}{\pi n}\right)-32\left(\frac{Cn}{a}\right)^2\gamma_2^2\left(\frac{n\pi}{a}\right)^2\left(\frac{\pi a}{n}\right)\Big\}\Big]$$

$$\frac{2W}{Ea^3} = \frac{1}{(1-\sigma^2)}\frac{\left(\frac{t}{a}\right)^3}{12}\frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2\Big[n^2\pi^2\left(1+16\gamma_2^2\right)+n^2\left(1+16\gamma_2^2\right)$$

$$+\beta_1^2+2n^2\left(1+16\gamma_2^2\right)+2n^2\left(\frac{\beta_1}{\pi n}\right)+2\frac{n^2}{\pi^2}\left(\frac{\beta_1}{\pi n}\right)$$

$$+(1-\sigma)\left\{2n^2\left(1+\frac{\beta_1}{\pi n}\right)^2+32\gamma_2^2 n^2-2n^2\left(1+\frac{\beta_1}{\pi n}\right)-32\gamma_2^2 n^2\right\}\Big]$$

$$(1-\sigma^2)\,\frac{2W_1}{Ea^3}\,\frac{1}{\left(\frac{t}{a}\right)} = \frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2\left(\frac{t}{a}\right)^2 n^2 \Big[(1+16\gamma_2^2)(\pi^2+3) + 4\left(\frac{\beta_1}{n}\right) + \left(\frac{\beta_1}{n}\right)^2 \qquad 51)$$

$$+ (1-\sigma)\left\{ 2\left(1+\frac{\beta_1}{n}\right)\left(\frac{\beta_1}{n}\right)\right\} \Big]$$

$$= \frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2\left(\frac{t}{a}\right)^2 n^2 \left[(\pi^2+3)(1+16\gamma_2^2) + \frac{\beta_1}{n}\left(4+\frac{\beta_1}{n}\right) + 2\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)(1-\sigma)\right]$$

<u>Total energy</u>

$$(1-\sigma^2)\,\frac{2W}{Ea^3}\,\frac{1}{\left(\frac{t}{a}\right)} = \frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2 \Big[\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right\}^2 + \frac{\beta_1^2}{n^2\pi^2} + \left(u_0 - \frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)(1+4\gamma_2^2)$$

$$+ \left(\frac{t}{a}\right)^2 n^2 \left\{ (1+16\gamma_2^2)(\pi^2+3) + \frac{\beta_1}{n}\left(4+\frac{\beta_1}{n}\right) + \overset{2(1-\sigma)}{\cancel{2(1-\sigma)}}\,\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)\right\}$$

$$- \left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1 + \frac{\beta_1}{\pi} + \frac{3}{4n\pi}\right)\gamma_2 + \frac{1}{\cancel{4}}\left(\frac{Cn}{a}\right)^2 \left\{ \left(\pi^2+\frac{1}{\pi^2}\right)\left(9\gamma_2^4 + \frac{17}{2}\gamma_2^2 + \frac{17}{32}\right) + \left(\frac{3}{2}\gamma_2^4 + 3\gamma_2^2 + \frac{1}{16}\right)\right\}$$

$$+ \frac{1-\sigma}{2}\left\{ \beta_1^2 + \frac{\alpha_1}{\pi}\left(\beta_1 + \frac{\alpha_1}{\pi} - \frac{1}{n}\right) + \frac{3}{2}\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n} + 2(1+4\gamma_2^2)\left(\frac{w_0}{a} - \frac{u_0}{\pi^2}\right)\right\}\Big]$$

$$+ 8(1-\sigma)\frac{\pi}{n}\,\frac{u_0 w_0}{a}$$

$$\frac{\left(\frac{t}{a}\right)^2 n^2}{12}\Big[\left(\pi+\frac{1}{\pi}\right)^2(1+16\gamma_2^2) + \frac{\beta_1}{n\pi}\left\{\frac{\beta_1}{n\pi} + 2\left(\pi+\frac{1}{\pi}\right)\right\}$$

$$+ 2(1-\sigma)\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)\Big]$$

$\frac{\partial W_c}{\partial \alpha_1} = 0$ gives

$$2\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right\} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 + \frac{1-\sigma}{2}\left\{\frac{1}{\pi}\left(\beta_1 + \frac{2\alpha_1}{\pi} + \frac{1}{n}\right)\right\} = 0$$

$\frac{\partial W_c}{\partial \beta_1} = 0$ gives

$$\frac{2}{\pi}\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right) - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\frac{\gamma_2}{\cancel{\pi}} + \frac{1-\sigma}{2}\left\{2\beta_1\pi + \frac{\alpha_1}{\pi}\right\} = 0$$

or $$\left(2 + \frac{1-\sigma}{2}\cdot\frac{2}{\pi^2}\right)\alpha_1 + \left(\frac{2}{\pi} + \frac{1-\sigma}{2}\frac{1}{\pi}\right)\beta_1 + \frac{2}{n\pi} + \frac{1-\sigma}{2}\frac{1}{n\pi} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 = 0.$$

or $$\left(\frac{2}{\pi} + \frac{1-\sigma}{2}\frac{1}{\pi}\right)\alpha_1 + \left(\frac{2}{\pi^2} + (1-\sigma)\right)\beta_1 + \frac{2}{n\pi^2} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\frac{\gamma_2}{\pi} = 0.$$

or $$\left(2 + \frac{1-\sigma}{\pi^2}\right)\alpha_1 + \frac{1}{\pi}\left(2 + \frac{1-\sigma}{2}\right)\beta_1 + \frac{1}{n\pi}\left(2 + \frac{1-\sigma}{2}\right) - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 = 0.$$

$$\left(2 + \frac{1-\sigma}{2}\right)\alpha_1 + \left[\frac{2}{\pi} + (1-\sigma)\pi\right]\beta_1 + \frac{2}{n\pi} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 = 0.$$

or $$\left(2 + \frac{1-\sigma}{2}\right)\alpha_1 + \frac{1}{\pi}\left[2 + (1-\sigma)\pi^2\right]\beta_1 + \frac{2}{n\pi} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 = 0.$$

$$(1-\sigma)\left(\frac{1}{\pi^2} - \frac{1}{2}\right)\alpha_1 + \frac{1}{\pi}\left[\frac{1}{2} - \frac{\pi^2}{\cancel{\pi^2}}\right)(1-\sigma)\beta_1 + \frac{1}{n\pi}\left[\frac{1-\sigma}{2}\right] = 0.$$

$$\begin{cases} \left(\frac{1}{2} - \frac{1}{\pi^2}\right)\alpha_1 + \frac{1}{\pi}\left(\pi^2 - \frac{1}{2}\right)\beta_1 - \frac{1}{2n\pi} = 0. \\ \left(2 + \frac{1-\sigma}{2}\right)\alpha_1 + \frac{1}{\pi}\left[2 + (1-\sigma)\pi^2\right]\beta_1 + \frac{2}{n\pi} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2 = 0 \end{cases}$$

$\frac{\partial W_e}{\partial u_0} = 0$ gives

$$\pi^2\left(\frac{Cn}{a}\right)^2\left[\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right) - \frac{1-\sigma}{2}\cdot\frac{2}{\pi^2}\cdot\left(1+4\gamma_2^2\right)\right] + 8(1-\sigma)\frac{w_0}{a} = 0$$

$$\pi^2\left(\frac{Cn}{a}\right)^2\left[\left(1+4\gamma_2^2\right)\left(1+\frac{\sigma}{\pi^2}\right)\right] + 8(1-\sigma)\frac{w_0}{a} = 0.$$

Thus
$$\boxed{\frac{w_0}{a} = (-)\;\frac{\pi^2\left(\frac{Cn}{a}\right)^2\left(1+\frac{\sigma}{\pi^2}\right)\left(1+4\gamma_2^2\right)}{8(1-\sigma)}}$$
X

$\frac{\partial W_e}{\partial w_0} = 0$ gives

$$\pi^2\left(\frac{Cn}{a}\right)^2\left[-\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right) + (1-\sigma)\left(1+4\gamma_2^2\right)\right] + 8(1-\sigma)u_0 = 0.$$

$$\boxed{u_0 = \frac{\pi^2\left(\frac{Cn}{a}\right)^2\left(1+4\gamma_2^2\right)\left(\frac{1}{\pi^2}+\sigma\right)}{8(1-\sigma)}}$$
O.K.

$$\frac{1}{\pi}\begin{vmatrix} \left(\frac{1}{2}-\frac{1}{\pi^2}\right) & \left(\pi^2-\frac{1}{2}\right) \\ 2+\frac{1-\sigma}{2} & 2+(1-\sigma)\pi^2 \end{vmatrix} = \left[\begin{matrix} 1+\cancel{\frac{(1-\sigma)\pi^2}{2}}-\frac{2}{\pi^2}-(1-\sigma) \\ -2\pi^2-\cancel{\frac{1-\sigma}{2}}+1+\frac{1-\sigma}{4} \end{matrix}\right]\frac{1}{\pi}$$

$$=\left[2\left(1-\pi^2-\frac{1}{\pi^2}\right)-\frac{3}{4}(1-\sigma)\right]\frac{1}{\pi}$$

$$\frac{1}{\pi}\begin{bmatrix} \frac{1}{2n\pi} & \left(\pi^2-\frac{1}{2}\right) \\ \left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\gamma_2-\frac{2}{n\pi} & 2+(1-\sigma)\pi^2 \end{bmatrix} = \frac{1}{\pi}\left[\begin{matrix} \frac{1}{n\pi}+\frac{(1-\sigma)\pi}{2n}+\frac{2}{n\pi}\left(\pi^2-\frac{1}{2}\right) \\ -\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)\gamma_2 \end{matrix}\right]$$

$$=\frac{1}{\pi}\left[\frac{(5-\sigma)\pi}{2n}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)\gamma_2\right]$$

$$\boxed{\alpha_1=\frac{\frac{(5-\sigma)\pi}{2n}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)\gamma_2}{2\left(1-\pi^2-\frac{1}{\pi^2}\right)-\frac{3}{4}(1-\sigma)}}$$

$$\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\left[\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\gamma_2-\frac{2}{n\pi}\right]-\frac{1}{2n\pi}\left(2+\frac{1-\sigma}{2}\right)$$

$$=\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\gamma_2-\frac{2}{n\pi}\left(\frac{1}{2}-\frac{1}{\pi^2}\right)-\frac{1}{n\pi}-\frac{1-\sigma}{4n\pi}$$

$$=\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\gamma_2\ \cancel{-\frac{1}{n\pi}}+\frac{1}{n\pi}\left\{\frac{2}{\pi^2}-\frac{(3-\sigma)}{4}\right\}$$

$$\boxed{\beta_1=\pi\,\frac{\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\gamma_2+\frac{1}{n\pi}\left\{\frac{2}{\pi^2}-\frac{(3-\sigma)}{4}\right\}}{2\left(1-\pi^2-\frac{1}{\pi^2}\right)-\frac{3}{4}(1-\sigma)}}$$

$$\frac{\partial W_2}{\partial \gamma_2} = 0 \quad \text{gives}$$

$$\frac{2\gamma_2}{n^2\pi^2} - \left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\left(\alpha_1 + \frac{\beta_1}{\pi} + \frac{3}{4n\pi}\right) + \left(u_0 - \frac{w_0}{a}\right)\left(1 + \frac{1}{\pi^2}\right) 8\gamma_2$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2 \left\{ \left(\pi^2 + \frac{1}{\pi^2}\right)\left(36\gamma_2^3 + 11\gamma_2\right) + \left(8\gamma_2^3 + 6\gamma_2\right) \right\}$$

$$+ \frac{1-\sigma}{2}\left\{ \frac{3}{2}\left(\frac{Cn}{a}\right)\frac{1}{n} + 16\gamma_2\left(\frac{w_0}{a} - \frac{u_0}{\pi^2}\right) \right\} = 0.$$

<u>But</u>

$$\alpha_1 + \frac{\beta_1}{\pi} = \frac{\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\gamma_2\left[\frac{1}{2} - \frac{1}{\pi^2} - \pi^2 + \frac{1}{2}\right] + \frac{1}{n\pi}\left\{\frac{2}{\pi^2} - \frac{(9-\sigma)}{4} + \frac{(5-\sigma)\pi^2}{2}\right\}}{2\left(1 - \pi^2 - \frac{1}{\pi^2}\right) - \frac{3}{4}(1-\sigma)}$$

$$= \frac{\left(\frac{Cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\left(\pi^2 + \frac{1}{\pi^2} - 1\right)\gamma_2 - \frac{1}{n\pi}\left\{2\left(\pi^2 + \frac{1}{\pi^2} - 1\right) + \frac{(1-\sigma)}{4}\left(2\pi^2 - 1\right)\right\}}{\frac{3}{4}(1-\sigma) + 2\left(\pi^2 + \frac{1}{\pi^2} - 1\right)}$$

$$\cancel{\left(u_0 - \frac{w_0}{a}\right) = \frac{1}{8(1-\sigma)}\pi^2\left(\frac{Cn}{a}\right)^2\left(1 + 4\gamma_2^2\right)\left[\frac{1}{\pi^2} + \sigma + 1 + \frac{\sigma}{\pi^2}\right]}$$

$$\cancel{= \frac{(1+\sigma)}{8(1-\sigma)}\pi^2\left(\frac{Cn}{a}\right)^2\left(1 + 4\gamma_2^2\right)\left(1 + \frac{1}{\pi^2}\right)}$$

$$\cancel{\left(\frac{w_0}{a} - \frac{u_0}{\pi^2}\right) = -\frac{1}{8(1-\sigma)}\pi^2\left(\frac{Cn}{a}\right)^2\left(1 + 4\gamma_2^2\right)\left[-1 + \frac{\sigma}{\pi^2} + \frac{1}{\pi^4} + \frac{\sigma}{\pi^2}\right]}$$

$$\cancel{= -\frac{1}{8(1-\sigma)}\pi^2\left(\frac{Cn}{a}\right)^2\left(1 + 4\gamma_2^2\right)\left[1 + \frac{1}{\pi^4} + \frac{2\sigma}{\pi^2}\right]}$$

Total potential energy of the system.

$$V = W - \sigma_{cr}\, u_0 \frac{2a}{n}\, 2\pi a t$$

$$= \frac{F}{2(1-\sigma^2)}\left(\frac{t}{a}\right)\cdot \mathcal{E}\, a^3 - \sigma_{cr}\, u_0 \frac{4\pi}{n}\cdot a^3\left(\frac{t}{a}\right)$$

$$= \frac{\pi}{n} a^3\left(\frac{t}{a}\right)\left[\frac{\mathcal{E} F}{2(1-\sigma^2)}\left(\frac{n}{\pi}\right) - \sigma_{cr}\, 4u_0\right]$$

$$= \frac{4\pi}{n} a^3\left(\frac{t}{a}\right)\left[\frac{\mathcal{E} F}{8(1-\sigma^2)}\left(\frac{n}{\pi}\right) - \sigma_{cr}\, u_0\right]$$

$$\frac{V}{\frac{4\pi}{n} a^3\left(\frac{t}{a}\right)} = \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\Bigg[\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right\}^2 + \frac{\beta_2^2}{n^2\pi^2} + \left(u_0 - \frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right)$$

$$+ \left(\frac{t}{a}\right)^2 n^2\left\{\left(1+16\gamma_2^2\right)\left(\pi^2+3\right) + \frac{\beta_1}{\pi n}\left(4+\frac{\beta_1}{n}\right) + 2(1-\sigma)\frac{\beta_1}{n}\left(1+\frac{\beta_1}{\pi n}\right)\right\}$$

$$-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{3}{4n\pi}\right)\gamma_2 + \frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi^2+\frac{1}{\pi^2}\right)\left(9\gamma_2^4+\frac{11}{2}\gamma_2^2+\frac{17}{32}\right)+\left(2\gamma_2^4+3\gamma_2^2+\frac{1}{16}\right)\right\}$$

$$+\frac{1-\sigma}{2}\left\{\beta_1^2 + \frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}+\frac{1}{n}\right) + \frac{3}{2}\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n} + 2\left(1+4\gamma_2^2\right)\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\right\}\Bigg]$$

$$+ \frac{1}{1+\sigma}\,\frac{u_0 w_0}{a} - \frac{\sigma_{cr}}{E}\, u_0$$

$\frac{\partial V}{\partial u_0}$ gives

$$\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\left\{\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right) - (1-\sigma)\left(1+4\gamma_2^2\right)\frac{1}{\pi^2}\right\} + \frac{1}{1+\sigma}\,\frac{w_0}{a} - \sigma_{cr} = 0$$

$$\sigma_{cr} - \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\left(1+4\gamma_2^2\right)\left(1+\frac{\sigma}{\pi^2}\right) = \frac{1}{1+\sigma}\,\frac{w_0}{a}$$

$$\boxed{\frac{w_0}{a} = (1+\sigma)\,\tilde{\sigma}_{cr} - \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}\left(1+\frac{\sigma}{\pi^2}\right)(1+4\gamma_2^2)}$$

$$u_0 - \frac{w_0}{a} = \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}(1+4\gamma_2^2)\left[\frac{1}{\pi^2}+\sigma+1+\frac{\sigma}{\pi^2}\right] - (1+\sigma)\,\tilde{\sigma}_{cr}$$

$$= (1+\sigma)\left\{\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}(1+4\gamma_2^2)\left(1+\frac{1}{\pi^2}\right) - \tilde{\sigma}_{cr}\right\}$$

$$\frac{w_0}{a} - \frac{u_0}{\pi^2} = (1+\sigma)\,\tilde{\sigma}_{cr} - \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}(1+4\gamma_2^2)\left[1+\frac{\sigma}{\pi^2}+\frac{1}{\pi^4}+\frac{\sigma}{\pi^2}\right]$$

$$= (1+\sigma)\,\tilde{\sigma}_{cr} - \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}(1+4\gamma_2^2)\left[1+\frac{1}{\pi^4}+\frac{2\sigma}{\pi^2}\right]$$

Put into the γ-equation, we have

$$\frac{2\gamma_2}{n^2\pi^2} - \left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left\{\frac{\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-1+\frac{1}{\pi^2}\right)\gamma_2^2 - \frac{1}{n\pi}\left\{\frac{1}{2}\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{1-\sigma}{4}\left(2\pi^2-\frac{13}{4}\right)\right\}}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\right\}$$

$$+ (1+\sigma)\left\{\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{(1-\sigma)}(1+4\gamma_2^2)\gamma_2\left(1+\frac{1}{\pi^2}\right) - \tilde{\sigma}_{cr}\cdot 8\gamma_2\right\}\left(1+\frac{1}{\pi^2}\right)$$

$$+ \frac{1}{2}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi^2+\frac{1}{\pi^2}\right)(36\gamma_2^3+17\gamma_2) + (8\gamma_2^3+6\gamma_2)\right\}$$

$$+ (1-\sigma)\frac{3}{4}\left(\frac{Cn}{a}\right)\frac{1}{n} + 8\gamma_2(1-\sigma^2)\,\tilde{\sigma}_{cr} - \pi^2\left(\frac{Cn}{a}\right)^2\gamma_2(1+4\gamma_2^2)\left[1+\frac{1}{\pi^4}+\frac{2\sigma}{\pi^2}\right] = 0.$$

$$\frac{2\gamma_2}{\pi^2} - 8\gamma_2 \left(\frac{\sigma_a n^2}{}\right)(1+\sigma)\left(\sigma + \frac{1}{\pi^2}\right)$$

$$+ \left(\frac{cn^2}{a}\right)\left\{ \frac{\left(1+\frac{1}{\pi^2}\right)\left[\frac{1}{2}\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{1-\sigma}{4}\left(2\pi^2 - \frac{13}{4}\right)\right]}{2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{3}{4}(1-\sigma)} + \frac{3}{4}(1-\sigma) \right\} \gamma_2$$

$$+ \left(\frac{cn^2}{a}\right)^2 \left\{ \left(\frac{1+\sigma}{1-\sigma}\right)\left(1+\frac{1}{\pi^2}\right)\pi^2(\gamma_2 + 4\gamma_2^3) - \frac{\left(\pi+\frac{1}{\pi}\right)^2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right)\gamma_2}{2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{3}{4}(1-\sigma)} \right.$$

$$\left. + \left(\pi^2 + \frac{1}{\pi^2}\right)\left(18\gamma_2^3 + \frac{11}{2}\gamma_2\right) + (4\gamma_2^3 + 3\gamma_2) - \pi^2\left[1 + \frac{1}{\pi^4} + \frac{2\sigma}{\pi^2}\right](\gamma_2 + 4\gamma_2^3) \right\} = 0$$

$$\left[\frac{2}{\pi^2} - 8(\sigma_a n^2)(1+\sigma)\left(\frac{1}{\pi^2} + \sigma\right)\right]\gamma_2 + \left(\frac{cn^2}{a}\right)\left\{\frac{3}{4}(1-\sigma) + \frac{\left(1+\frac{1}{\pi^2}\right)\left[\frac{1}{2}\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{1-\sigma}{4}\left(2\pi^2 - \frac{13}{4}\right)\right]}{2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{3}{4}(1-\sigma)}\right\}$$

$$+ \left(\frac{cn^2}{a}\right)^2 \left\{ \left[\frac{2\sigma}{1-\sigma}\left(\pi + \frac{1}{\pi}\right)^2 + 2(1-\sigma)\right](\gamma_2 + 4\gamma_2^3) - \frac{\left(\pi+\frac{1}{\pi}\right)^2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right)\gamma_2}{2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{3}{4}(1-\sigma)} \right.$$

$$\left. + \left(\pi^2 + \frac{1}{\pi^2}\right)\left(18\gamma_2^3 + \frac{11}{2}\gamma_2\right) + (4\gamma_2^3 + 3\gamma_2) \right\} - 0.32\left(\frac{tn^2}{a}\right)^2(\pi^2+3)\gamma_2 = 0.$$

$\equiv$

for $\left(\frac{cn^2}{a}\right)^2$ a factor $\frac{\left(\pi + \frac{1}{\pi}\right)^2}{12(\pi^2+3)} = 0.077515$

$\frac{\partial V}{\partial\left(\frac{c}{a}\right)} = 0$ gives

$$2\left\{\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right\}^2 + \frac{2\gamma_2^2}{n^2\pi^2} + 2\left(u_0 - \frac{w_0}{a}\right)\left(1 + \frac{1}{\pi^2}\right)(1 + 4\gamma_2^2)$$

$$+ 2\left(\frac{t}{a}\right)^2 n^2 \left\{(1 + 16\gamma_2^2)(\pi^2+3) + \frac{\beta_1}{n}\left(4 + \frac{\beta_1}{n}\right) + 2(1-\sigma)\frac{\beta_1}{n}\left(1 + \frac{\beta_1}{n}\right)\right\}$$

$$- 3\left(\frac{cn}{a}\right)\left(\pi + \frac{1}{\pi}\right)\left(\alpha_1 + \frac{\beta_1}{\pi} + \frac{3}{4n\pi}\right)\gamma_2 + 2\left(\frac{cn}{a}\right)^2\left\{\left(\pi^2 + \frac{1}{\pi^2}\right)\left(9\gamma_2^4 + \frac{11}{2}\gamma_2^2 + \frac{17}{32}\right) + 2\gamma_2^4 + 3\gamma_2^2 + \frac{1}{16}\right\}$$

$$+ (1-\sigma)\left\{\beta_1^2 + \frac{\alpha_1}{\pi}\left(\beta_1 + \frac{\alpha_1}{\pi} + \frac{1}{n}\right) + \frac{7}{4}\left(\frac{cn}{a}\right)\frac{\gamma_2}{n} + 2(1 + 4\gamma_2^2)\left(\frac{w_0}{a} - \frac{u_0}{\pi^2}\right)\right\} = 0.$$

$$4\left\{\frac{\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-1+\frac{1}{\pi^2}\right)\zeta_2-\frac{1}{\pi}\frac{1-\sigma}{2}\left(\pi^2-2\right)}{2\left(\pi^2+\frac{1}{\pi^2}-1\right)+\frac{3}{4}(1-\sigma)}\right\}^2+\frac{4\zeta_2^2}{\pi^2}$$

$$+4(1+\sigma)\left(1+\frac{1}{\pi^2}\right)(1+4\zeta_2^2)\left\{\frac{\pi^2\left(\frac{Cn^2}{a}\right)^2}{8(1-\sigma)}(1+4\zeta_2^2)\left(1+\frac{1}{\pi^2}\right)-\left(\sigma_{cr}\cdot n^2\right)\right\}$$

$$+4\left(\frac{tn^2}{a}\right)^2\left\{(1+16\zeta_2^2)(3+\pi^2)+\frac{(3-2\sigma)}{n^4}\pi^2\left[\frac{\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\zeta_2+\frac{1}{4\pi}\left\{\frac{2}{\pi^2}-\frac{(9-\sigma)}{4}\right\}}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\right]^2\right.$$

$$\left.-\frac{2(3-\sigma)}{n^2}\pi\,\frac{\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\zeta_2+\frac{1}{4\pi}\left\{\frac{2}{\pi^2}-\frac{(9-\sigma)}{4}\right\}}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\right\}$$

$$-\frac{3}{2}\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\zeta_2\,\frac{\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-1+\frac{1}{\pi^2}\right)\zeta_2-\frac{1}{\pi}\left\{\frac{1}{2}\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{1-\sigma}{4}\left(2\pi^2-\frac{13}{4}\right)\right\}}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}$$

$$+4\left(\frac{Cn^2}{a}\right)^2\left\{\left(\pi^2+\frac{1}{\pi^2}\right)\left(9\zeta_2^4+\frac{11}{2}\zeta_2^2+\frac{17}{32}\right)+\left(2\zeta_2^4+3\zeta_2^2+\frac{1}{16}\right)\right\}$$

$$+\frac{(1-\sigma)}{2}\left\{\pi^2\left[\frac{\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\zeta_2+\frac{1}{\pi}\left\{\frac{2}{\pi^2}-\frac{9-\sigma}{4}\right\}}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\right]^2\right.$$

$$+\frac{1}{\pi}\left[\frac{\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)\zeta_2-\frac{(5-\sigma)}{2}\pi}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\right]\left[\frac{\frac{1}{2}\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)^2\zeta_2+2(\pi^2-1)+\frac{1-\sigma}{2}}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\right]$$

$$\left.+\frac{9}{4}\left(\frac{Cn^2}{a}\right)\zeta_2+2(1+4\zeta_2^2)(1+\sigma)\left(\sigma_{cr}n^2\right)-\frac{\pi^2\left(\frac{Cn^2}{a}\right)^2}{4(1-\sigma)}(1+4\zeta_2)^2\left[1+\frac{1}{\pi^4}+\frac{2\sigma}{\pi^2}\right]\right\}=0.$$

With $\sigma = 0.3$. 60)

$$\frac{1}{\pi^2} = 0.10132$$

$$\frac{1}{\pi^2} + \sigma = 0.40132$$

$$1+\sigma = 1.3000$$

$$\frac{3}{4}(1-\sigma) = \frac{3}{4} \times 0.700 = 0.52500$$

$$2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{3}{4}(1-\sigma) = 17.94184 + 0.52500 = 18.46684$$

$$\left(1+\frac{1}{\pi^2}\right)\left[\frac{1}{2}\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) + \frac{1-\sigma}{4}\left(2\pi^2 - \frac{13}{4}\right)\right]$$

$$= 1.10132\left[4.48546 + \frac{0.7}{4} \times 16.48920\right] = 1.10132 \times 7.37107$$

$$= 8.11791$$

$$\frac{2\sigma}{1-\sigma}\left(\pi + \frac{1}{\pi}\right)^2 + 2(1-\sigma) = \frac{0.6}{0.7}\left[9.86960 + 2 + 0.10132\right] + 1.4000$$

$$= 10.26079 + 1.4000 = 11.66079$$

$$\left(\pi + \frac{1}{\pi}\right)^2\left(\pi^2 - 1 + \frac{1}{\pi^2}\right) = 11.97092 \times 8.97092 = 107.39017$$

$$\left(\pi^2 + \frac{1}{\pi^2}\right) = 9.97092 \qquad \pi^2 + 3 = 12.86960$$

Thus
$$\left\{0.20264 - 4.17373(\sigma n^2)\right\}\gamma_2 + \left(\frac{Cn}{a}\right)^8 \times 0.96459$$

$$+ \left(\frac{Cn}{a}\right)^2\left\{11.66079\left(\gamma_2 + 4\gamma_2^3\right) - 5.81530\,\gamma_2 + 9.97092\left(18\gamma_2^3 + 5.5\gamma_2\right)\right.$$

$$\left. + \left(4\gamma_2^3 + 3\gamma_2\right)\right\} + \left(\frac{tn^2}{a}\right)^2\gamma_2 \cdot 411.8272 = 0$$

$$\left\{411.8272\left(\frac{tn^2}{a}\right)^2+0.20262-4.17373\left(\frac{\sigma n^2}{E}\right)\right\}\zeta_2$$

$$+\ 0.96459\left(\frac{Cn^2}{a}\right)+\ 63.68555\,\zeta_2\left(\frac{Cn^2}{a}\right)^2+\ 230.11972\,\zeta_2^3\left(\frac{Cn^2}{a}\right)^2=0.$$

$$\cancel{\text{Let } \zeta_2 = x,\quad \left(\frac{Cn^2}{a}\right)=y}$$

$$\cancel{A\zeta_2+B}$$

$$\cancel{\boxed{Ax+By+C\,xy^2+D\,x^2y^2=0}}$$

$$\left(1.68077\left(\frac{Cn^2}{a}\right)\zeta_2-0.04748\right)^2+0.20264\,\zeta_2^2$$

$$+\ 1.43172\,(1+4\zeta_2^2)\left[1.94100\left(\frac{Cn^2}{a}\right)^2(1+4\zeta_2^2)-\left(\frac{\sigma_a}{E}n^2\right)\right]$$

$$+\left(\frac{tn^2}{a}\right)^2\left\{12.86960\,(1+16\zeta_2^2)+\frac{2.4}{n^4}\pi^2\left[0.074695\left(\frac{Cn^2}{a}\right)\zeta_2-0.033997\right]^2\right.$$

$$\left.-\ \frac{5.4}{n^2}\pi\left[0.074695\left(\frac{Cn^2}{a}\right)\zeta_2-0.033997\right]\right\}$$

$$-\ 5.18984\left(\frac{Cn^2}{a}\right)\zeta_2\left[1.68074\left(\frac{Cn^2}{a}\right)\zeta_2-0.12705\right]$$

$$+\left(\frac{Cn^2}{a}\right)^2\left\{9.97092\left(9\zeta_2^4+5.5\,\zeta_2^2+\frac{17}{32}\right)+\left(2\zeta_2^4+3\zeta_2^2+\frac{1}{16}\right)\right\}$$

$$+0.35\left\{\pi^2\left[0.074695\left(\frac{Cn^2}{a}\right)\zeta_2-0.033997\right]^2+\left[0.55878\left(\frac{Cn^2}{a}\right)\zeta_2-0.12725\right]\right.$$

$$\cdot\left[0.324119\left(\frac{Cn^2}{a}\right)\zeta_2+0.979550\right]+2.25\left(\frac{Cn^2}{a}\right)\zeta_2+2.6\left(\frac{\sigma_a}{E}n^2\right)(1+4\zeta_2^2)$$

$$\left.-\ 3.77533\left(\frac{Cn^2}{a}\right)^2(1+4\zeta_2^2)^2\right\}=0.$$

$$\gamma_2^2\Big[0.20264 + (1.68077)^2\left(\frac{Cn^2}{a}\right)^2 + 11.66080\left(\frac{Cn^2}{a}\right)^2 - 2.08688\left(\frac{\sigma}{E}n^2\right)$$

$$+ 12.86960\times 16\left(\frac{tn^2}{a}\right)^2 - 8.72293\left(\frac{Cn^2}{a}\right)^2 + 57.84006\left(\frac{Cn^2}{a}\right)^2$$

$$+ 0.35\,(\pi\times 0.074695)^2\left(\frac{Cn^2}{a}\right)^2 + 0.35\times 0.55878\times 0.324119\left(\frac{Cn^2}{a}\right)^2\Big]$$

$$+\gamma_2^4\Big[91.73828\left(\frac{Cn^2}{a}\right)^2 + \overset{23.32160}{\cancel{5.83040}}\left(\frac{Cn^2}{a}\right)^2\Big]$$

$$+\gamma_2\Big[-1.68077\times 0.09496\left(\frac{Cn^2}{a}\right) + 5.18984\times 0.12705\left(\frac{Cn^2}{a}\right)$$

$$-0.7\pi^2\times 0.074695\times 0.033997\left(\frac{Cn^2}{a}\right) + 0.35\,(0.979550\times 0.55878 - 0.12726\times 0.324119)\left(\frac{Cn^2}{a}\right)$$

$$+ 0.35\times 2.25\left(\frac{Cn^2}{a}\right)\Big]$$

$$+\Big[(0.04748)^2 + 1.45760\left(\frac{Cn^2}{a}\right)^2 - 0.52172\left(\frac{\sigma}{E}n^2\right) + 12.86960\left(\frac{tn^2}{a}\right)$$

$$+ 5.35955\left(\frac{Cn^2}{a}\right)^2 + 0.35\,(\pi\times 0.033997)^2 + 0.35\times 0.979550\times 0.12726\Big] = 0.$$

$$\Big[\overset{A}{115.05988\left(\frac{Cn^2}{a}\right)^2}\Big]\gamma_2^4 + \Big[63.68558\left(\frac{Cn^2}{a}\right)^2 \overset{+0.20264\ \ B}{+ \underline{205.9136\left(\frac{tn^2}{a}\right)^2}} - 2.08688\left(\frac{\sigma}{E}n^2\right)\Big]\gamma_2^2$$

$$+\Big[\overset{C}{1.446891}\ \frac{Cn^2}{a}\Big]\gamma_2 + \Big[\overset{-0.037384}{\cancel{0.002254}} + 6.81715\left(\frac{Cn^2}{a}\right)^2 \overset{D}{+ \underline{12.86960\left(\frac{tn}{a}\right)^2}} - 0.52172\left(\frac{\sigma}{E}n^2\right)\Big]$$

$$= 0.$$

$$\Big[\cancel{63.68555}$$

$$\Big[\overset{E}{230.11972\left(\frac{Cn^2}{a}\right)^2}\Big]\gamma_2^3 + \Big[63.68555\left(\frac{Cn^2}{a}\right)^2 + 0.20264 \overset{F}{+ \underline{411.8272\left(\frac{tn^2}{a}\right)^2}} - 4.17373\left(\frac{\sigma}{E}n^2\right)\Big]\gamma_2$$

$$+ \overset{G}{0.96459}\left(\frac{Cn^2}{a}\right) = 0$$

(3)

$Ax^4 + Bx^2 + Cx + D = 0$

$Ex^3 + Fx + G = 0.$

$Ax^6 + 0 + Bx^4 + Cx^3 + Dx^2 + 0 + 0 = 0$

$0 + Ax^5 + 0 + Bx^3 + Cx^2 + Dx + 0 = 0$

$0 + 0 + Ax^4 + 0 + Bx^2 + Cx + D = 0$

$Ex^6 + 0 + Fx^4 + Gx^3 + 0 + 0 + 0 = 0$

$0 + Ex^5 + 0 + Fx^3 + Gx^2 + 0 + 0 = 0$

$0 + 0 + Ex^4 + 0 + Fx^2 + Gx + 0 = 0$

$0 + 0 + 0 + Ex^3 + 0 + Fx + G = 0$

A	0	B	C	D	0	0
0	A	0	B	C	D	0
0	0	A	0	B	C	D
E	0	F	G	0	0	0
0	E	0	F	G	0	0
0	0	E	0	F	G	0
0	0	0	E	0	F	G

$$A\cdot\begin{vmatrix} A & 0 & B & C & D & 0 \\ 0 & A & 0 & B & C & D \\ 0 & F & G & 0 & 0 & 0 \\ E & 0 & F & G & 0 & 0 \\ 0 & E & 0 & F & G & 0 \\ 0 & 0 & E & 0 & F & G \end{vmatrix} - E\begin{vmatrix} 0 & B & C & D & 0 & 0 \\ A & 0 & B & C & D & 0 \\ 0 & A & 0 & B & C & D \\ E & 0 & F & G & 0 & 0 \\ 0 & E & 0 & F & G & 0 \\ 0 & 0 & E & 0 & F & G \end{vmatrix}$$

$$= -A\cdot F\begin{vmatrix} A & B & C & D & 0 \\ 0 & 0 & B & C & D \\ E & F & G & 0 & 0 \\ 0 & 0 & F & G & 0 \\ 0 & E & 0 & F & G \end{vmatrix} + A\cdot G\begin{vmatrix} A & 0 & C & D & 0 \\ 0 & A & B & C & D \\ E & 0 & G & 0 & 0 \\ 0 & E & F & G & 0 \\ 0 & 0 & 0 & F & G \end{vmatrix}$$

$$+ ED\begin{vmatrix} 0 & B & C & D & 0 \\ A & 0 & B & C & D \\ E & 0 & F & G & 0 \\ 0 & E & 0 & F & G \\ 0 & 0 & E & 0 & F \end{vmatrix} - EG\begin{vmatrix} 0 & B & C & D & 0 \\ A & 0 & B & C & D \\ 0 & A & 0 & B & C \\ E & 0 & F & G & 0 \\ 0 & E & 0 & F & G \end{vmatrix}$$

$$= ADF\begin{vmatrix} A & B & C & D \\ E & F & G & 0 \\ 0 & 0 & F & G \\ 0 & E & 0 & F \end{vmatrix} - AFG\begin{vmatrix} A & B & C & D \\ 0 & 0 & B & C \\ E & F & G & 0 \\ 0 & 0 & F & G \end{vmatrix} - ADG\begin{vmatrix} A & 0 & C & D \\ E & 0 & G & 0 \\ 0 & E & F & G \\ 0 & 0 & 0 & F \end{vmatrix} + A\cdot G^2\begin{vmatrix} A & 0 & C & D \\ 0 & A & B & C \\ E & 0 & G & 0 \\ 0 & E & F & G \end{vmatrix}$$

$$+ DE^2\begin{vmatrix} 0 & B & D & 0 \\ A & 0 & C & D \\ E & 0 & G & 0 \\ 0 & E & F & G \end{vmatrix} + DEF\begin{vmatrix} 0 & B & C & D \\ A & 0 & B & C \\ E & 0 & F & G \\ 0 & E & 0 & F \end{vmatrix} + AEG\begin{vmatrix} B & C & D & 0 \\ A & 0 & B & C \\ 0 & F & G & 0 \\ E & 0 & F & G \end{vmatrix} + E^2G\begin{vmatrix} B & C & D & 0 \\ 0 & B & C & D \\ A & 0 & B & C \\ 0 & F & G & 0 \end{vmatrix}$$

65)

$$= ADEF\begin{vmatrix} A & C & D \\ E & G & 0 \\ 0 & F & G \end{vmatrix} + ADF^2\begin{vmatrix} A & B & C \\ E & F & G \\ 0 & 0 & F \end{vmatrix} + AF^2G\begin{vmatrix} A & B & D \\ 0 & 0 & C \\ E & F & 0 \end{vmatrix}$$

$$- AFG^2\begin{vmatrix} A & B & C \\ 0 & 0 & B \\ E & F & G \end{vmatrix} - ADFG\begin{vmatrix} A & 0 & C \\ E & 0 & G \\ 0 & E & F \end{vmatrix} + AEG^2\begin{vmatrix} 0 & C & D \\ A & B & C \\ E & F & G \end{vmatrix} + AG^3\begin{vmatrix} A & 0 & D \\ 0 & A & C \\ 0 & E & G \end{vmatrix}$$

$$- BDE^2\begin{vmatrix} A & C & D \\ E & G & 0 \\ 0 & F & G \end{vmatrix} + D^2E^2\begin{vmatrix} A & 0 & D \\ E & 0 & 0 \\ 0 & E & G \end{vmatrix} + DE^2F\begin{vmatrix} 0 & C & D \\ A & B & C \\ E & F & G \end{vmatrix} + DEF^2\begin{vmatrix} 0 & B & C \\ A & 0 & B \\ E & 0 & F \end{vmatrix}$$

$$+ ACEG\begin{vmatrix} B & C & D \\ 0 & F & G \\ E & 0 & F \end{vmatrix} + AEG^2\begin{vmatrix} B & C & D \\ A & 0 & B \\ 0 & F & G \end{vmatrix} + DE^2G\begin{vmatrix} B & C & D \\ A & 0 & B \\ 0 & F & G \end{vmatrix} - CE^2G\begin{vmatrix} B & C & D \\ 0 & E & C \\ 0 & F & G \end{vmatrix}$$

$$= ADEF(AG^2 + DEF - CEG) + ADF^3(AF - BE) - ACF^2G(AF - BE)$$

$$+ ABFG^2(AF - BE) + ADEFG(AG - CE) + AEG^2(ADF + C^2E - BDE - ACG)$$

$$+ A^2G^3(AG - CE) - BDE^2(AG^2 + DEF - CEG) - D^3E^4$$

$$+ DE^2F(C^2E + ADF - BDE - ACG) - BDEF^2(AF - BE)$$

$$+ ACEG(BF^2 + CEG - DEF) + AEG^2(ADF - ACG - B^2F)$$

$$+ DE^2G(ADF - ACG - B^2F) - BCE^2G(BG - CF)$$

$$= 4A^2DEFG^2 + 2AD^2E^2F^2 - 4ACDE^2FG + A^2DF^4 - 2ABDEF^3$$

$$- A^2CF^3G + 2ABCEF^2G + A^2BF^2G^2 - 2AB^2EFG^2 + A^2DEFG^2$$

$$- ACDE^2FG + A^2DEFG^2 + AC^2E^2G^2 - 2ABDE^2G^2 - 3A^2CEG^3$$

$$+ A^3G^4 - A^2CEG^3 - ABDE^2G^2 - 2BD^2E^3F + BCDE^3G$$

$$- D^3E^4 + C^2DE^3F + AD^2E^2F^2 - BD^2E^3F - ACDE^2FG$$

$$- ABDEF^3 + B^2DE^2F^2 + ABCEF^2G + AC^2E^2G^2 - ACDE^2FG$$

$$+ A^2DEFG^2 - A^2CEG^3 - AB^2EFG^2 + AD^2E^2FG$$

$$- ACDE^2G^2 - B^2DE^2FG - B^2CE^2G^2 - BC^2E^2FG$$

$$= 4A^2DEFG^2 + 2AD^2E^2F^2 - 4ACDE^2FG + A^2DF^4 - 2ABDEF^3$$

$$- A^2CF^3G + 2ABCEF^2G + A^2BF^2G^2 - 2AB^2EFG^2$$

$$+ AC^2E^2G^2 - 2ABDE^2G^2 - 3A^2CEG^3 + A^3G^4 - 2BD^2E^3F$$

$$+ BCDE^3G - D^3E^4 + C^2DE^3F + B^2DE^2F^2 + AC^2E^2G^2$$

$$+ AD^2E^2FG - ACDE^2G^2 - B^2DE^2FG - B^2CE^2G^2$$

$$- BC^2E^2FG = 0$$

67)

$$\cancel{A^2}\ 2AC^2E^2G^2 - 3A^2CEG^3 + A^3G^4 - D^3E^4 + \cancel{ACDEG} - ACDE^2G^2$$

$$- B^2CE^2G^2$$

$$= 2 \times 115.060 \times (1.44689)^2 \times (230.1197)^2 \times (0.96459)^2 \times \left(\frac{Cn^2}{a}\right)^{10}$$

$$- 3(115.060)^2 \times 1.44689 \times 230.1197 \times (0.96459)^3 \times \left(\frac{Cn^2}{a}\right)^{10}$$

$$+ (115.060)^3 \times (0.96459)^4 \left(\frac{Cn^2}{a}\right)^{10}$$

$$- (230.1197)^4 \left(\frac{Cn^2}{a}\right)^8 \left[12.8696\left(\frac{tn^2}{a}\right) + 6.81715\left(\frac{Cn^2}{a}\right)^2 - 0.037384 - 0.52172\left(\frac{\sigma n^2}{E}\right)\right]^3$$

$$- (115.060)(1.44689)(230.1197)^2(0.96459)^2\left[\frac{Cn^2}{a}\right]^9\left[12.8696\left(\frac{tn^2}{a}\right) + 6.81715\left(\frac{Cn^2}{a}\right)^2 - 0.037384 - 0.52172\left(\frac{\sigma n^2}{E}\right)\right]$$

$$- (1.44689)(230.1197)^2(0.96459)^2\left(\frac{Cn^2}{a}\right)^7\left[63.68558\left(\frac{Cn^2}{a}\right)^2 + 205.9136\left(\frac{tn^2}{a}\right)^2 + 0.20264 - 2.08688\,\frac{\sigma n^2}{E}\right]^2$$

$$= (230.1197)^2\left(\frac{Cn^2}{a}\right)^7\Bigg\{\Bigg[230.120 \times 1.44689^2 \times 0.96459^2 - \frac{3 \times 115.060 \times 1.44689 \times 0.96459^3}{230.1197}$$

$$+ 115.060 \times \left(\frac{115.060}{230.1197}\right)^2 \times (0.96459)^4\Bigg]\left(\frac{Cn^2}{a}\right)^3$$

$$-\Bigg[(230.1197)^2\left(\frac{Cn^2}{a}\right)\left\{12.8696\left(\frac{tn^2}{a}\right)^2 + 6.81715\left(\frac{Cn^2}{a}\right)^2 - 0.037384 - 0.52172\left(\frac{\sigma n^2}{E}\right)\right\}^2$$

$$+ 115.060)(1.44689)(0.96459)^2\left(\frac{Cn^2}{a}\right)^2\Bigg]\left[12.8696\left(\frac{tn^2}{a}\right)^2 + 6.81715\left(\frac{Cn^2}{a}\right)^2 - 0.037384 - 0.52172\left(\frac{\sigma n^2}{E}\right)\right]$$

$$- (1.44689)(0.96459)^2\left[63.68558\left(\frac{Cn^2}{a}\right)^2 + 205.9136\left(\frac{tn^2}{a}\right)^2 + 0.20264 - 2.08688\,\frac{\sigma n^2}{E}\right]^2\Bigg\}$$

$$DF\left[4A^2EG^2 - 4ACE^2G + C^2E^3\right]$$ 68)

$$\left[12.8696\left(\frac{tn^2}{a}\right)^2 + 6.81715\left(\frac{Cn^2}{a}\right)^2 - 0.52172\left(\frac{\sigma n^2}{E}\right) - 0.037384\right]$$

$$\left[63.68555\left(\frac{Cn^2}{a}\right)^2 + 0.20264 + 411.8272\left(\frac{tn^2}{a}\right)^2 - 4.17373\left(\frac{\sigma n^2}{E}\right)\right]$$

$$\overset{8}{}$$

$$\left[4(115.060)^2\left(\overset{230.1197}{\cancel{1.44689}}\right)(0.96459)^2 - 4(115.060)(1.44689)(230.1197)^2(0.96459)\right.$$

$$\left. + (1.44689)^2(230.1197)^3\right]\left(\frac{Cn^2}{a}\right)^8$$

$$B^2DE^2F^2 - B^2DE^2FG = B^2DF\left[E^2F - E^2G\right] = E^2B^2DF\left[F-G\right]$$

$$= (230.1197)^2\left(\frac{Cn^2}{a}\right)^4\left[63.68558\left(\frac{Cn^2}{a}\right)^2 + 205.9136\left(\frac{tn^2}{a}\right)^2 + 0.20264 - 2.08688\left(\frac{\sigma n^2}{E}\right)\right]^2$$

$$\left[12.8696\left(\frac{tn^2}{a}\right)^2 + 6.81715\left(\frac{Cn^2}{a}\right)^2 - 0.037384 - 0.52172\left(\frac{\sigma n^2}{E}\right)\right]^2$$

$$\left[63.68558\left(\frac{Cn^2}{a}\right)^2 + 411.8272\left(\frac{tn^2}{a}\right)^2 + 0.20264 - 4.17373\left(\frac{\sigma n^2}{E}\right)\right]$$

$$\left[63.68555\left(\frac{Cn^2}{a}\right)^2 \quad 0.96459\left(\frac{Cn^2}{a}\right) + 411.8272\left(\frac{tn^2}{a}\right)^2 + 0.20264 - 4.17373\left(\frac{\sigma n^2}{E}\right)\right]$$

Put $\frac{l(Cn^2)}{2}\frac{x}{l_2} = X$, $\frac{y}{2} = y$

$$0 = 115.060\, X^2 y^2 + 63.6856\, X^2 + \left[0.20264 + 205.914\left(\frac{tn^2}{a}\right)^2 - 2.0863\,\frac{\sigma n^2}{E}\right] y^2$$

$$+ 1.4769\, X + 6.8132\,\frac{X^2}{y^2} + \left[12.8696\left(\frac{tn^2}{a}\right)^2 - 0.52122\left(\frac{\sigma n^2}{E}\right) - 0.037384\right]$$

$$0 = 230.120\, X^2 y^2 + 65.6856\, X^2 + \left[0.20264 + 411.827\left(\frac{tn^2}{a}\right)^2 - 4.1737\left(\frac{\sigma n^2}{E}\right)\right] y^2$$

$$+ 0.96459\, X$$

$$y^2 = \frac{-X\left[0.96459 + 63.6856\, X\right]}{230.120\, X^2 + \left[0.20264 + 411.827\left(\frac{tn^2}{a}\right)^2 - 4.1737\left(\frac{\sigma n^2}{E}\right)\right]}$$

$$0 = -\frac{X\left[0.96459 + 63.6856\, X\right]\left[115.060\, X^2 + 0.20264 + 205.914\left(\frac{tn^2}{a}\right)^2 - 2.0863\left(\frac{\sigma n^2}{E}\right)\right]}{230.120\, X^2 + \left[0.20264 + 411.827\left(\frac{tn^2}{a}\right)^2 - 4.1737\left(\frac{\sigma n^2}{E}\right)\right]}$$

$$+ 63.6856\, X^2 + 1.4769\, X - \frac{6.8132\, X\left[230.120\, X^2 + 0.20264 + 411.827\left(\frac{tn^2}{a}\right)^2 - 4.1737\frac{\sigma n^2}{E}\right]}{\left[0.96459 + 63.6856\, X\right]}$$

$$+ \left[12.8696\left(\frac{tn^2}{a}\right)^2 - 0.52122\left(\frac{\sigma n^2}{E}\right) - 0.037384\right] = 0.$$

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20264
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$$-x\left[0.96459+63.6856x\right]^2\left[115.060x^2+0.20264+205.914\left(\frac{tn^2}{a}\right)^2\underline{-2.0169\left(\frac{\sigma n^2}{E}\right)}\right]$$

$$+\left[0.96459+63.6856x\right]\left[230.120x^2+0.20264+411.827\left(\frac{tn^2}{a}\right)^2\underline{-4.1737\left(\frac{\sigma n^2}{E}\right)}\right]\left[\begin{array}{l}63.6856x^2+1.4469x\\ +12.8696\left(\frac{tn^2}{a}\right)^2\underline{-0.52172\left(\frac{\sigma n^2}{E}\right)}\\ -0.037384\end{array}\right]$$

$$-6.8172x\left[230.120x^2+0.20264+411.827\left(\frac{tn^2}{a}\right)^2\underline{-4.1737\left(\frac{\sigma n^2}{E}\right)}\right]^2=0$$

$$\left[(0.96459+63.6856x)\,4.1737\times0.52172-6.8172x\,(4.1737)^2\right]\left(\frac{\sigma n^2}{E}\right)^2$$

$$+\Big[x(0.96459+63.6856x)^2\,2.0169-(0.96459+63.6856x)\Big\{4.1737\Big(63.6856x^2+1.4469x+12.8696\left(\frac{tn^2}{a}\right)^2-0.037384\Big)$$

$$+0.52172\Big(230.120x^2+0.20264+411.827\left(\frac{tn^2}{a}\right)^2\Big)\Big\}+13.6344\times4.1737x\Big\{230.120x^2+0.20264+411.827\left(\frac{tn^2}{a}\right)^2\Big\}\Big]\left(\frac{\sigma n^2}{E}\right)$$

$$+\left[0.96459+63.6856x\right]\Big\{\left[230.120x^2+0.20264+411.127\left(\frac{tn^2}{a}\right)^2\right]\left[63.6856x^2+1.4469x+12.8696\left(\frac{tn^2}{a}\right)^2-0.037384\right]$$

$$-x\left[0.96457+63.6856x\right]\left[115.060x^2+0.20264+205.914\left(\frac{tn^2}{a}\right)^2\right]\Big\}$$

$$-6.8172x\left[230.110x^2+0.20264+411.827\left(\frac{tn^2}{a}\right)^2\right]^2=0$$

(8)

$$4.1237\left[0.50325+4.7732x\right]=A$$

$$B=2.0869x(0.96459+63.6856x)^2-(0.96459+63.6856x)\left[385.863x^2+6.03893x-0.05031+268.572\left(\frac{tn^2}{a}\right)^2\right]+56.9059x\left[230.120x^2+0.20264+411.827\left(\frac{tn^2}{a}\right)^2\right]$$

$$=2.0869x\left(0.96459^2+1.92918\times 63.6856x+63.6856^2x^2\right)$$

$$-(0.96459+63.6856x)(385.863x^2+6.03893x-0.05031)$$

$$+56.9059x(230.120x^2+0.20264)$$

$$+(23.5250x-0.96459)\,268.572\left(\frac{tn^2}{a}\right)^2$$

$$=(-3014.03x^3-500.394x^2+10.8520x+0.04853)+268.572(23.5250x-0.96459)\left(\frac{tn^2}{a}\right)^2$$

$$C = [0.96459 + 63.6856x]$$

$$\Big[(230.120x^2 + 0.20264)(63.6856x^2 + 1.4469x - 0.037384) - \frac{x[0.96459 + 63.6856x]}{[115.060x^2 + 0.20264]}$$

$$+ \{411.827(63.6856x^2 + 1.4469x - 0.037384) - 205.914x(0.96459 + 63.6856x)$$

$$+ 12.8696(230.120x^2 + 0.20264)\}\left(\frac{tn^2}{a}\right)^2 + 411.827 \times 12.8696\left(\frac{tn^2}{a}\right)^4\Big]$$

$$- 6.8172x(230.120x^2 + 0.20264)^2 - 13.6344 \times 411.827x[230.120x^2 + 0.20264]\left(\frac{tn^2}{a}\right)^2$$

$$- (411.827)^2 \times 6.8172x\left(\frac{tn^2}{a}\right)^4$$

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$$= [0.96459 + 63.6856x]$$

$$\Big[7327.66x^4 + 221.976x^3 - 8.6028x^2 + 0.09773x - 0.007575$$

$$+ (15438.39x^2 + 387.604x - 12.7878)\left(\frac{tn^2}{a}\right)^2 + 5300.05\left(\frac{tn^2}{a}\right)^4\Big]$$

$$- 6.8172x(52955.21x^4 + 93.26303x^2 + 0.041063)$$

$$- 5615.014x(230.120x^2 + 0.20264)\left(\frac{tn^2}{a}\right)^2 - 1{,}156{,}207.35x\left(\frac{tn^2}{a}\right)^4$$

$$= 105660.2x^5 + 21204.87x^4 - 969.552x^3 - 2.07418x^2 - 0.66808x - 0.007307$$

$$- (308923.9x^3 + 39576.51x^2 + 1578.346x + 12.33498)\left(\frac{tn^2}{a}\right)^2$$

$$- (818670.5x - 5112.375)\left(\frac{tn^2}{a}\right)^4$$

Putting $\quad 10x = \xi$ $\hfill$ 73)

$$10\left(\frac{tn^2}{a}\right) = \eta$$

$$A = 1.9922\,\xi + 2.1004$$

$$B = \left(\overset{0.490711}{\cancel{6.3316}}\,\xi - \overset{0.20071}{\cancel{2.5906}}\right)\eta^2 - \left(3.0140\,\xi^3 + 5.0039\,\xi^2 - 1.0652\,\xi + 0.04453\right)$$

$$C = 1.0566\,\xi^5 + 2.1205\,\xi^4 - 0.9696\,\xi^3 - 0.02074\,\xi^2 - 0.06681\,\xi - 0.007307$$
$$-\left(\overset{0.33346}{\cancel{3.0172}}\,\xi^3 - \overset{0.31178}{3.9577}\,\xi^2 + \overset{0.12234}{1.5783}\,\xi + \overset{0.0095614}{0.12335}\right)\eta^2$$
$$-\left(\overset{0.049190}{\cancel{8.1112}}\,\xi - \overset{0.0030718}{\cancel{0.5104}}\right)\eta^4$$

Take $\quad \frac{t}{a} = \frac{1}{1000}, \quad n = 10 \quad \eta = 1$

$\xi = 0$

Probable $\frac{\sigma n^2}{E} \approx \underline{\underline{0.03}}$

$$A = 2.1004$$
$$B = -2.6391$$
$$C = +0.38058$$

$$\frac{\sigma n^2}{E} = \frac{1}{4.2008}\left[2.6391 \pm \sqrt{(2.6391)^2 - 4 \times 2.1004 \times 0.38058}\right]$$

$$= \frac{1}{2.1004}\left[1.3196 \pm \sqrt{(1.3196)^2 - 2.1004 \times 0.38058}\right]$$

$$= \frac{1}{2.1004}\left[1.3196 \pm 0.9705\right] = \frac{0.1662}{1.0903}$$

74)

The condition $y^2 \geq 0$, gives

$$\cancel{-x[0.096459 + 0.636856x]}$$

$$\frac{-\xi[0.096459 + 0.636856\xi]}{2.30120\xi^2 + 0.20264 + 0.31923\eta - 4.1737\left(\frac{\sigma n^2}{E}\right)} \geq 0$$

Take $\frac{t}{a} = \frac{1}{1000}$ $\quad n = 10 \quad$ $\eta = 1$, then

$$A = 1.9922\xi + 2.1004$$

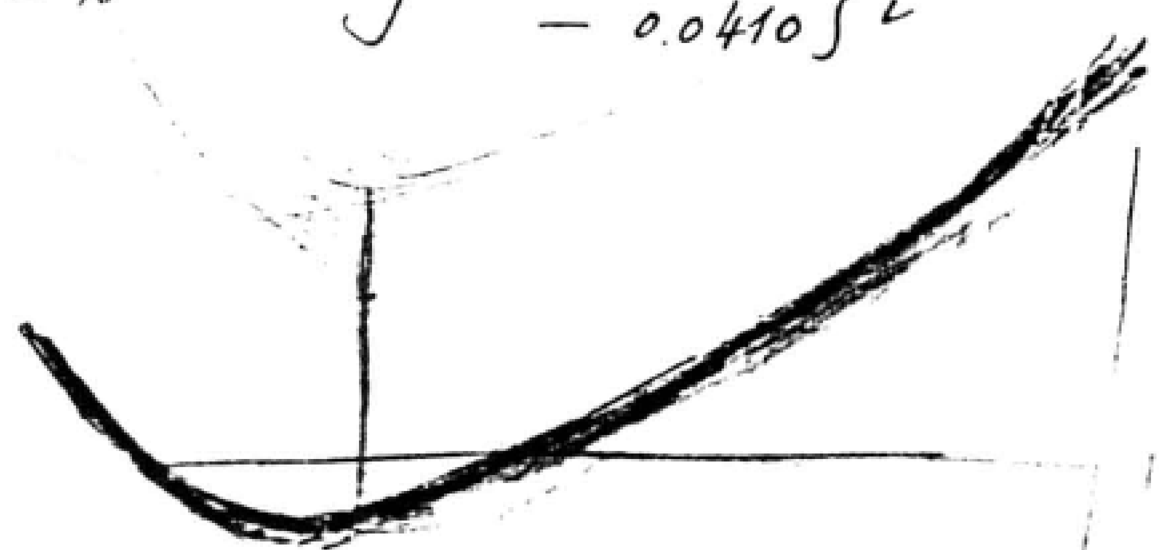
$$B = -(3.0140\xi^3 + 5.0039\xi^2 - 1.5760\xi + 0.24934)$$

$$C = 1.0566\xi^5 + 2.1205\xi^4 - 1.2091\xi^3 + 0.28604\xi^2 - 0.23834\xi - 0.013796$$

for $\xi = 0$,

$$\frac{\sigma n^2}{E} = \frac{1}{2.1004}\left[+0.12467 \pm \sqrt{(0.12469)^2 + 2.1004 \times 0.013796}\right]$$

$$= \frac{1}{2.1004}\left[0.12467 \pm \sqrt{0.044520}\right] = \begin{cases} +0.1598 \\ -0.0410 \end{cases}$$



$\xi = 1$

$$A = 4.0926$$

$$B = -6.6912$$

$$C = 2.0019$$

$$\frac{\sigma n^2}{E} = \frac{1}{4.0926}\left[+3.3456 \pm \sqrt{(3.3456)^2 - 4.0926 \times 2.0019}\right]$$

$$= \begin{matrix} +0.39425 \\ +1.24070 \end{matrix}$$

75)

$$\xi = -1$$

$$A = 0.1082$$

$$B = -1^{x}\ 3.8152$$

$$C = 2.7835$$

$$m^{2} = \frac{1}{0.1082}\left[+1.9076 \pm \sqrt{1.9076 - 0.1082 \times 2.7835}\right]$$

$$= \frac{1}{0.1082}\left[1.9076 \pm \sqrt{3.3377}\right]$$

$$= \begin{cases} +0.627 \\ 34.634 \end{cases}$$

Correction Part. to Energy (7b)

$$(k_1+k_2)^2 = \left\{\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial v}{\partial\theta}\right)\right\}^2$$

$$= C^2\left\{\left(\frac{n\pi}{a}\right)^2 \sin n\theta \sin\frac{n\pi x}{a} + 4\gamma_2\left(\frac{n\pi}{a}\right)^2 \cos 2n\theta\cos\frac{2n\pi}{a}\right.$$

$$\left.+\left(\frac{n}{a}\right)^2\left[\left(1+\frac{\beta_1}{n}\right)\sin n\theta\sin\frac{n\pi x}{a} + 4\gamma_2\cos 2n\theta\cos\frac{2n\pi}{a}\right]\right\}^2$$

$$= \left(\frac{Cn}{a}\right)^2\left(\frac{n}{a}\right)^2\frac{\pi a}{n}\left\{\left(\pi^2+1+\frac{\beta_1}{n}\right)^2+16\gamma_2^2(\pi^2+1)^2\right\}$$

$$k_1k_2-\tau^2 = \frac{1}{a^2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial\theta^2}+\frac{\partial v}{\partial\theta}\right) - \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x\partial\theta}+\frac{\partial v}{\partial x}\right)^2$$

$$= \left(\frac{Cn}{a}\right)^2\left(\frac{n}{a}\right)^2\left[\sin n\theta\sin\frac{n\pi x}{a}+4\gamma_2\cos 2n\theta\cos\frac{2n\pi x}{a}\right]$$

$$\left[\left(1+\frac{\beta_1}{n}\right)\sin n\theta\sin\frac{n\pi x}{a}+4\gamma_2\cos 2n\theta\cos\frac{2n\pi x}{a}\right]\pi^2$$

$$-\left(\frac{Cn}{a}\right)^2\left(\frac{n}{a}\right)^2\pi^2\left[\left(1+\frac{\beta_1}{n}\right)\cos n\theta\cos\frac{n\pi x}{a}+4\gamma_2\sin 2n\theta\sin\frac{2n\pi x}{a}\right]^2$$

$$= \left(\frac{Cn}{a}\right)^2\left(\frac{n}{a}\right)^2\pi^2\left[\left(1+\frac{\beta_1}{n}\right)+\cancel{16\gamma_2^2}-\left(1+\frac{\beta_1}{n}\right)^2-\cancel{16\gamma_2^2}\right]\frac{\pi a}{n}$$

$$\frac{2W_b}{\pi a} = \frac{t^3}{12}\left(\frac{Cn}{a}\right)^2\left(\frac{n}{a}\right)^2\pi^2\left(\frac{\pi a}{n}\right)\left[\left(\pi+\frac{1}{\pi}+\frac{\beta_1}{n\pi}\right)^2+16\gamma_2^2\left(\pi+\frac{1}{\pi}\right)^2\right.$$

$$\left.-2(1-\sigma)\left\{\left(1+\frac{\beta_1}{n}\right)\left(-1-1-\frac{\beta_1}{n}\right)\right\}\right]$$

$$= \frac{t^3}{12(1-\sigma^2)}\left(\frac{Cn}{a}\right)^2\left(\frac{n}{a}\right)^2\pi^2\left(\frac{\pi a}{n}\right)\left[\left(\pi+\frac{1}{\pi}\right)^2\left(1+16\gamma_2^2\right)+\left(\frac{\beta_1}{n\pi}\right)^2+2\frac{\beta_1}{n\pi}\left(\pi+\frac{1}{\pi}\right)+2(1-\sigma)\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)\right]$$

$$\frac{{}^{3}_{2}V_{b}^{5}}{\lambda^{3}} = \left(\frac{C_{n}}{a}\right)^{2}\left(\frac{t}{a}\right)^{3} n^{2}\left(\frac{\pi^{3}}{\varkappa}\right)\frac{1}{12(1-\sigma^{2})}\Bigg[\left(\pi+\frac{1}{\pi}\right)^{2}\left(1+16\gamma_{2}^{2}\right)+\left(\frac{\beta_{1}}{n\pi}\right)\left\{\frac{\beta_{1}}{n\pi}+2\left(\pi+\frac{1}{\pi}\right)\right\}$$

$$+\,2(1-\sigma)\frac{\beta_{1}}{n}\left(1+\frac{\beta_{1}}{n}\right)\Bigg] \qquad 77)$$

$$\iint (\epsilon_1+\epsilon_2)^2\, dx\, d\theta$$

$$= \left(\frac{Cn}{a}^2\right)\frac{\pi a}{n}\,\pi^2\Bigg[\left\{\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right\}^2+\left\{\frac{\gamma_2}{n\pi}\right\}^2-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left[\alpha_1+\frac{\beta_1}{n\pi}+\frac{3}{4n\pi}\right]\gamma_2$$

$$+\frac{1}{4}\left(\frac{Cn}{a}^2\right)\left\{\left(\pi+\frac{1}{\pi}\right)^2\left[4\left(\gamma_2^2+\frac{1}{4}\right)^2+2\gamma_2^2+\gamma_2^4+\frac{1}{16}\right]+\left(\pi-\frac{1}{\pi}\right)^2\left[\frac{1}{4}+2\gamma_2^2+4\gamma_2^4\right]\right\}\Bigg]$$

$$=\left(\frac{Cn}{a}^2\right)^2\frac{\pi a}{n}\,\pi^2\Bigg[\left\{\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right\}^2+\frac{\gamma_2^2}{n^2\pi^2}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{n\pi}+\frac{3}{4n\pi}\right)\gamma_2$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^4+4\gamma_2^2+\frac{5}{16}\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^4+2\gamma_2^2+\frac{1}{4}\right)\right\}$$

Total strain energy

$$(1-\sigma^2)\frac{2W}{Ea^3}\frac{1}{\left(\frac{t}{a}\right)}=\frac{\pi^3}{n}\left(\frac{Cn}{a}\right)^2\Bigg[\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)^2+\frac{\gamma_2^2}{n^2\pi^2}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{n\pi}+\frac{3}{4n\pi}\right)\gamma_2$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^4+4\gamma_2^2+\frac{5}{16}\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^4+2\gamma_2^2+\frac{1}{4}\right)\right\}$$

$$+\frac{1-\sigma}{2}\left\{\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}-\frac{1}{n}\right)+2\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\left(1+4\gamma_2^2\right)\right\}$$

$$+\frac{1-\sigma}{4}\,3\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n}+\left(u_0-\frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right)$$

$$+\frac{\left(\frac{tn}{a}\right)^2}{12}\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(1+16\gamma_2^2\right)+\frac{\beta_1}{n\pi}\left[\frac{\beta_1}{n\pi}+2\left(\pi+\frac{1}{\pi}\right)\right]+2(1-\sigma)\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)\right\}\Bigg]$$

$$+8(1-\sigma)\frac{\pi}{n}\frac{u_0 w_0}{a}$$

Total potential energy of the system

$$\frac{V}{\frac{4\pi}{n}a^3\left(\frac{t}{a}\right)} = \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\Bigg[\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)^2+\frac{\gamma_2^2}{n^2\pi^2}-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{n\pi}+\frac{3}{4n\pi}\right)\gamma_2$$

$$+\left(u_0-\frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right)+\frac{1-\sigma}{2}\left\{\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}-\frac{1}{n}\right)+2\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\left(1+4\gamma_2^2\right)\right\}$$

$$+(1-\sigma)\frac{3}{4}\left(\frac{Cn}{a}\right)\frac{\gamma_2}{n}+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^4+4\gamma_2^2+\frac{5}{16}\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^4+1\gamma_2^2+\frac{1}{4}\right)\right\}$$

$$+\frac{\left(\frac{tn}{a}\right)^2}{12}\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(1+16\gamma_2^2\right)+\frac{\beta_1}{n\pi}\left[\frac{\beta_1}{n\pi}+2\left(\pi+\frac{1}{\pi}\right)\right]+2(1-\sigma)\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)\right\}\Bigg]$$

$$+\frac{1}{1+\sigma}\frac{u_0w_0}{a}-\left(\frac{\tilde{\sigma}}{E}\right)u_0$$

$$=\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\Bigg[\left\{\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)^2+\frac{\gamma_2^2}{n^2\pi^2}+\frac{1-\sigma}{2}\left[\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}-\frac{1}{n}\right)\right]\right.$$

$$+\left(1+4\gamma_2^2\right)\left[\left(u_0-\frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)+(1-\sigma)\left(\frac{w_0}{a}-\frac{u_0}{\pi^2}\right)\right]$$

$$+\cancel{\left(u_0-\frac{w_0}{a}\right)\left(1+\frac{1}{\pi^2}\right)\left(1+4\gamma_2^2\right)}$$

$$* \quad -\left(\frac{Cn}{a}\right)\left\{\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{n\pi}+\frac{3}{4n\pi}\right)-(1-\sigma)\frac{3}{4}\frac{1}{n}\right\}\gamma_2$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^4+4\gamma_2^2+\frac{5}{16}\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^4+2\gamma_2^2+\frac{1}{4}\right)\right\}$$

$$+\frac{\left(\frac{tn}{a}\right)^2}{12}\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(1+16\gamma_2^2\right)+\frac{\beta_1}{n\pi}\left[\frac{\beta_1}{n\pi}+2\left(\pi+\frac{1}{\pi}\right)\right]+2(1-\sigma)\frac{\beta_1}{n}\left(1+\frac{\beta_1}{n}\right)\right\}\Bigg]$$

$$+\frac{1}{1+\sigma}\frac{u_0w_0}{a}-\left(\frac{\sigma}{E}\right)u_0$$

$$\frac{V}{\left(\frac{4\pi}{n}\right)a^3\left(\frac{t}{a}\right)} = \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\Bigg[\Big\{\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)^2+\frac{1-\sigma}{2}\left[\beta_1^2+\frac{\alpha_1}{\pi}\left(\beta_1+\frac{\alpha_1}{\pi}-\frac{1}{n}\right)\right]+\frac{\gamma_2^2}{n^2\pi^2}$$

$$+\left(1+4\gamma_2^2\right)\left[u_0\left(1+\frac{\sigma}{\pi^2}\right)-\frac{w_0}{a}\left(\sigma+\frac{1}{\pi^2}\right)\right]\Big\}$$

$$-\left(\frac{Cn}{a}\right)\gamma_2\left[\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{n\pi}\right)+\frac{3}{4n}\left(\sigma+\frac{1}{\pi^2}\right)\right]$$

$$+\frac{1}{4}\left(\frac{Cn}{a}\right)^2\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^4+4\gamma_2^2+\frac{5}{16}\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^4+2\gamma_2^2+\frac{1}{4}\right)\right\}$$

$$+\frac{\left(\frac{tn}{a}\right)^2}{12}\left\{\left(\pi+\frac{1}{\pi}\right)^2\left(1+16\gamma_2^2\right)+\left(2-2\sigma+\frac{1}{\pi^2}\right)\left(\frac{\beta_1}{n}\right)^2+2\left(2+\frac{1}{\pi^2}-\sigma\right)\left(\frac{\beta_1}{n}\right)\right\}\Bigg]$$

$$+\frac{1}{1+\sigma}\frac{u_0 w_0}{a}-\left(\frac{\sigma}{E}\right)u_0$$

$\frac{\partial V}{\partial \alpha_1}=0$ gives

$$2\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)+\frac{1-\sigma}{2\pi}\left(\beta_1+\frac{2\alpha_1}{\pi}-\frac{1}{n}\right)$$

$$-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)=0$$

$$\left(2+\frac{1-\sigma}{\pi^2}\right)\alpha_1+\frac{1}{\pi}\left(2+\frac{1-\sigma}{2}\right)\beta_1+\frac{1}{n\pi}\left(2-\frac{1-\sigma}{2}\right)-\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\gamma_2=0$$

$\frac{\partial V}{\partial \beta_1}=0$ gives

$$2\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)\frac{1}{\pi}+\frac{1-\sigma}{2}\left[2\beta_1+\frac{\alpha_1}{\pi}\right]-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)\frac{1}{n\pi}=0.$$

$$2\left(\alpha_1+\frac{\beta_1}{\pi}+\frac{1}{n\pi}\right)+\frac{1-\sigma}{2}\left[2\pi\beta_1+\alpha_1\right]-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)=0.$$

81)

$$\left(2+\frac{1-\sigma}{2}\right)\alpha_1+\left[\frac{2}{\pi}+(1-\sigma)\pi\right]\beta_1+\frac{2}{n\pi}-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)=0$$

$$\left(2+\frac{1-\sigma}{2}\right)\alpha_1+\frac{1}{\pi}\left[2+(1-\sigma)\pi^2\right]\beta_1+\frac{2}{n\pi}-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)=0.$$

$$\left(2+\frac{1-\sigma}{\pi^2}\right)\alpha_1+\frac{1}{\pi}\left[2+\frac{1-\sigma}{2}\right]\beta_1+\frac{1}{n\pi}\left(2-\frac{1-\sigma}{2}\right)-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)=0$$

$$(1-\sigma)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\alpha_1+\frac{1}{\pi}(1-\sigma)\left(\pi^2-\frac{1}{2}\right)\beta_1+\frac{(1-\sigma)}{2n\pi}=0.$$

or

$$\begin{cases}\left(\frac{1}{2}-\frac{1}{\pi^2}\right)\alpha_1+\frac{1}{\pi}\left(\pi^2-\frac{1}{2}\right)\beta_1+\frac{1}{2n\pi}=0.\\ \left(2+\frac{1-\sigma}{2}\right)\alpha_1+\frac{1}{\pi}\left[2+(1-\sigma)\pi^2\right]\beta_1+\left[\frac{2}{n\pi}-\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)\right]=0.\end{cases}$$

The denominator

$$\frac{1}{\pi}\left[1+\cancel{\frac{(1-\sigma)\pi^2}{2}}-\frac{2}{\pi^2}-(1-\sigma)-2\pi^2+1-\cancel{\frac{(1-\sigma)\pi^2}{2}}+\frac{1-\sigma}{4}\right]$$

$$=\frac{1}{\pi}\left[2\left(1-\pi^2-\frac{1}{\pi^2}\right)-\frac{3}{4}(1-\sigma)\right]$$

$$=-\frac{1}{\pi}\left[\frac{3}{4}(1-\sigma)+2\left(\pi^2-1+\frac{1}{\pi^2}\right)\right]$$

The numerator for α_1

$$-\frac{1}{\pi}\left[\frac{1}{n\pi}+\frac{(1-\sigma)\pi}{2n}-\frac{2\pi}{n}+\frac{1}{n\pi}+\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)\pi^2-\left(\frac{Cn}{a}\right)\gamma_2\frac{\pi+\frac{1}{\pi}}{2}\right]$$

$$=-\frac{1}{\pi}\left[\frac{2}{n\pi}-\frac{2\pi}{n}+\frac{(1-\sigma)\pi}{2n}+\left(\frac{Cn}{a}\right)\gamma_2\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)\right]$$

$$=-\frac{1}{\pi}\left[\frac{2}{n}\left(\frac{1}{\pi}-\pi\right)+\frac{(1-\sigma)\pi}{2n}+\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)\gamma_2\right]$$

Therefore

$$\boxed{\alpha_1 = \frac{\frac{2}{n}\left(\frac{1}{\pi}-\pi\right)+\frac{(1-\sigma)\pi}{2n}+\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-\frac{1}{2}\right)f_2}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}}$$

The numerator for β_1

$$-\left[\cancel{\frac{1}{n\pi}}-\frac{2}{n\pi^3}\left(\frac{Cn}{a}\right)f_2\left(\pi+\frac{1}{\pi}\right)\left(\frac{1}{2}-\frac{1}{\pi^2}\right)-\cancel{\frac{1}{n\pi}}-\frac{(1-\sigma)}{4n\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{2}{n\pi^2}+\frac{1-\sigma}{4n}+\left(\frac{Cn}{a}\right)f_2\left(\pi+\frac{1}{\pi}\right)\left(\frac{\pi}{2}-\frac{1}{\pi}\right)\right]$$

$$\boxed{\beta_1 = -\frac{\frac{2}{n\pi^2}+\frac{1-\sigma}{4n}+\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{\pi}{2}-\frac{1}{\pi}\right)f_2}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}}$$

for $\frac{\partial V}{\partial u_0}$ gives

$$\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\left(1+\frac{\sigma}{\pi^2}\right)+\frac{1}{1+\sigma}\frac{w_0}{a}-\frac{\sigma}{E}=0$$

or $$\boxed{\frac{w_0}{a}=(1+\sigma)\left\{\frac{\sigma}{E}-\frac{\pi^2\left(\frac{Cn}{a}\right)^2(1+4f_2^2)}{8(1-\sigma^2)}\left(1+\frac{\sigma}{\pi^2}\right)\right\}}$$

for $\frac{\partial V}{\partial w_0}$ gives

$$-\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma^2)}\left(\sigma+\frac{1}{\pi^2}\right)+\frac{1}{1+\sigma}u_0=0$$

$$\boxed{u_0=\frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}\left(\sigma+\frac{1}{\pi^2}\right)(1+4f_2^2)}$$

$\frac{\partial V}{\partial \gamma_2} = 0$ gives

$$\frac{2\gamma_2}{n^2\pi^2} - \left(\frac{Cn}{a}\right)\left[\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{\pi}\right)+\frac{3}{4n}\left(\sigma+\frac{1}{\pi^2}\right)\right]$$

$$+ \left(\frac{Cn}{a}\right)^2\left[\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^3+2\gamma_2\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^3+\gamma_2\right)\right]$$

$$+ \frac{\left(\frac{tn}{a}\right)^2}{12} \times 32 \left(\pi+\frac{1}{\pi}\right)^2\gamma_2 \cancel{+0} + 8\gamma_2\left[u_0\left(1+\frac{\sigma}{\pi^2}\right)-\frac{w_0}{a}\left(\sigma+\frac{1}{\pi^2}\right)\right] = 0.$$

$$\frac{2\gamma_2}{n^2\pi^2} - \left(\frac{Cn}{a}\right)\left[\left(\pi+\frac{1}{\pi}\right)\left(\alpha_1+\frac{\beta_1}{\pi}\right)+\frac{3}{4n}\left(\sigma+\frac{1}{\pi^2}\right)\right]+\left(\frac{Cn}{a}\right)^2\left[\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^3+2\gamma_2\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^3+\gamma_2\right)\right]$$

$$+ \frac{8}{3}\left(\frac{tn}{a}\right)^2\left(\pi+\frac{1}{\pi}\right)^2\gamma_2 + 8\gamma_2\left[u_0\left(1+\frac{\sigma}{\pi^2}\right)-\frac{w_0}{a}\left(\sigma+\frac{1}{\pi^2}\right)\right] = 0.$$

Using the previous result;

$$\alpha_1 + \frac{\beta_1}{\pi} = \frac{\frac{2}{n}\left(\frac{1}{\pi}-\pi\right)+\frac{(1-\sigma)\pi}{2n}-\frac{2}{n\pi^3}-\frac{1-\sigma}{4n\pi}+\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\gamma_2\left[\pi^2-\frac{1}{2}-\frac{1}{2}+\frac{1}{\pi^2}\right]}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}$$

$$= \frac{\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\pi^2-1+\frac{1}{\pi^2}\right)\gamma_2 - \frac{1}{n}\left[\frac{2}{\pi}\left(\pi^2-1+\frac{1}{\pi^2}\right)-\frac{(1-\sigma)}{2}\left(\pi-\frac{1}{2\pi}\right)\right]}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}$$

Also

$$u_0\left(1+\frac{\sigma}{\pi^2}\right)-\frac{w_0}{a}\left(\sigma+\frac{1}{\pi^2}\right)$$

$$= \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}\left(1+4\gamma_2^2\right)\left[\left(1+\frac{\sigma}{\pi^2}\right)\left(\sigma+\frac{1}{\pi^2}\right)+\left(1+\frac{\sigma}{\pi^2}\right)\left(\sigma+\frac{1}{\pi^2}\right)\right] - \frac{\sigma}{E}(1+\sigma)\left(\sigma+\frac{1}{\pi^2}\right)$$

$$= \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}\left(1+4\gamma_2^2\right)\left[2\left(1+\frac{\sigma}{\pi^2}\right)\left(\sigma+\frac{1}{\pi^2}\right)\right] - \frac{\sigma}{E}(1+\sigma)\left(\sigma+\frac{1}{\pi^2}\right)$$

$$\frac{2\gamma_2}{\pi^2} - \frac{\left(\frac{Cn^2}{a}\right)^2\left(\pi+\frac{1}{\pi}\right)^2\left(\pi^2-1+\frac{1}{\pi^2}\right)}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}\gamma_2 + \left(\frac{Cn^2}{a}\right)\left[\frac{\left\{\frac{2}{\pi}\left(\pi^2-1+\frac{1}{\pi^2}\right)-\frac{1-\sigma}{2}\left(\pi-\frac{1}{2\pi}\right)\right\}\left(\pi+\frac{1}{\pi}\right)}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)} - \frac{3}{4}\left(\sigma+\frac{1}{\pi^2}\right)\right]$$

$$+\left(\frac{Cn^2}{a}\right)^2\left[\left(\pi+\frac{1}{\pi}\right)^2\left(5\gamma_2^3+2\gamma_2\right)+\left(\pi-\frac{1}{\pi}\right)^2\left(4\gamma_2^3+\gamma_2\right)\right]$$

$$+\frac{8}{3}\left(\frac{tn^2}{a}\right)^2\left(\pi+\frac{1}{\pi}\right)^2\gamma_2 + \frac{2\pi^2}{(1-\sigma)}\left(\frac{Cn^2}{a}\right)^2\left(\gamma_2+4\gamma_2^3\right)\left(1+\frac{\sigma}{\pi^2}\right)\left(\sigma+\frac{1}{\pi^2}\right) - 8(1+\sigma)\left(\sigma+\frac{1}{\pi^2}\right)\left(\frac{\sigma n^2}{E}\right)\gamma_2 = 0$$

Putting in the numerical values, $\sigma = 0.3000$

$$\frac{1}{\pi^2} = 0.10132$$

$$\varphi = 2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma) = 2\times 8.97092 + 0.525 = 18.46684$$

$$\psi = \left(\pi+\frac{1}{\pi}\right)^2\left(\pi^2-1+\frac{1}{\pi^2}\right) = \pi^2\left(1+\frac{1}{\pi^2}\right)^2\left(\pi^2-1+\frac{1}{\pi^2}\right)$$

$$= 9.86960\times 1.21291\times 8.97092 = 107.3903$$

$$\frac{\psi}{\varphi} = 5.81532$$

$$\frac{2}{\pi}\left(\pi^2-1+\frac{1}{\pi^2}\right)-\frac{1-\sigma}{2}\left(\pi-\frac{1}{2\pi}\right) = \pi\left[\frac{2}{\pi^2}\left(\pi^2-1+\frac{1}{\pi^2}\right)-\frac{1-\sigma}{2}\left(1-\frac{1}{2\pi^2}\right)\right]$$

$$= \pi\left[0.20264\times 8.97092 - 0.35\times 0.94934\right]$$

$$= \pi\left[1.48560\right] = f$$

$$\left\{\begin{array}{l} \left(\pi+\frac{1}{\pi}\right)f = \pi^2\cdot 1.10132\times 1.48560 = 16.14785 \\ -\frac{3}{4}\left(\sigma+\frac{1}{\pi^2}\right) = 0.75\times 0.40132 = 0.30099 \end{array}\right.$$

~~1.84616~~

$$\frac{16.14785}{18.46684} = 0.87443$$

$$0.87443 - 0.30099 = 0.57344$$

$$(\pi+\frac{1}{\pi})^2 = \pi^2 \times 1.10132^2 = \pi^2 \times 1.21291 = 11.97094$$

$$(\pi-\frac{1}{\pi})^2 = \pi^2 \times 0.89868^2 = \pi^2 \times 0.80763 = \cancel{7.97096}\ 7.97099$$

$$\frac{8}{3}(\pi+\frac{1}{\pi})^2 = 31.92295$$

$$\frac{2\pi^2}{(1-\sigma)}(1+\frac{\sigma}{\pi^2})(\sigma+\frac{1}{\pi^2}) = \frac{2(\pi^2+\sigma)(\sigma+\frac{1}{\pi^2})}{(1-\sigma)} = \frac{2\times 10.16960 \times 0.40132}{0.70}$$

$$= 8.16253/0.70 = 11.66076$$

$$0.20264\, f_2 - 5.81532\left(\frac{Cn^2}{a}\right)^2 f_2 + 0.57344\left(\frac{Cn^2}{a}\right)$$

$$+\left(\frac{Cn^2}{a}\right)^2\left[11.97094\,(5f_2^3+2f_2) + 7.97099\,(4f_2^3+f_2)\right]$$

$$+\ 31.92295\left(\frac{tn^2}{a}\right)^2 f_2 + \overset{11.66076}{\cancel{8.16253}}\left(\frac{Cn^2}{a}\right)^2(f_2+4f_2^3) - 0.417373\left(\frac{\sigma n^2}{E}\right)f_2 = 0.$$

$$\overset{138.382}{\cancel{124.389}}\left(\frac{Cn^2}{a}\right)^2 f_2^3 + \left[0.20264 + \overset{43.57363}{\cancel{40.07540}}\left(\frac{Cn}{a}\right)^2 + 31.92295\left(\frac{tn^2}{a}\right)^2 - 4.17373\left(\frac{\sigma n^2}{E}\right)\right] f_2$$

$$+\ 0.57344\left(\frac{Cn^2}{a}\right) = 0.$$

Putting $\left(\frac{Cn^2}{a}\right)f_2 = x$

$$\overset{138.382}{\cancel{124.389}}\, x^2 f_2^2 + \left[0.20264 + \overset{\cancel{43.57363}}{31.92295}\left(\frac{tn^2}{a}\right)^2 - 4.17373\left(\frac{\sigma n^2}{E}\right)\right] f_2^2$$

$$+\ \underset{43.57363}{\cancel{40.07540}}\, x^2 + 0.57344\, x = 0.$$

$$f_2^2 = -\frac{x\,(0.57344 + 40.07540\, x)}{124.389\, x^2 + 0.20264 + 31.92295\left(\frac{tn^2}{a}\right)^2 - 4.17373\left(\frac{\sigma n^2}{E}\right)}$$

Let $\quad 10x = \xi \qquad 10\left(\frac{tn^2}{a}\right) = \eta$

$$\boxed{f_2^2 = -\frac{\xi\,(0.057344 + 0.\overset{43574}{\cancel{400754}}\,\xi)}{\cancel{1.24389}\ \xi^2 + 0.20264 + 0.31923\,\eta^2 - 4.17373\left(\frac{\sigma n^2}{E}\right)}}$$

$$\frac{\partial V}{\partial(\frac{C}{a})} = 0 \quad \text{gives}$$ 86)

$$2\left\{ (\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi})^2 + \frac{1-\sigma}{2}\left[\beta_1^2 + \frac{\alpha_1}{\pi}(\beta_1 + \frac{\alpha_1}{\pi} - \frac{1}{n})\right] + \frac{\gamma_2^2}{n^2\pi^2} + (1+4\gamma_2^2)\left[u_0(1+\frac{\sigma}{\pi^2}) - \frac{w_0}{a}(\sigma+\frac{1}{\pi^2})\right]\right\}$$

$$- 3\left(\frac{Cn}{a}\right)\gamma_2\left[(\pi+\frac{1}{\pi})(\alpha_1 + \frac{\beta_1}{\pi}) + \frac{3}{4n}(\sigma + \frac{1}{\pi^2})\right]$$

$$+ \left(\frac{Cn^2}{a}\right)\left\{ (\pi+\frac{1}{\pi})^2(5\gamma_2^4 + 4\gamma_2^2 + \frac{5}{16}) + (\pi-\frac{1}{\pi})^2(4\gamma_2^4 + 2\gamma_2^2 + \frac{1}{4})\right\}$$

$$+ \frac{\left(\frac{tn}{a}\right)^2}{6}\left\{ (\pi+\frac{1}{\pi})^2(1+16\gamma_2^2)\right\} = 0.$$

$$\left\{ (\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi})^2 + \frac{1-\sigma}{2}\left[\beta_1^2 + \frac{\alpha_1}{\pi}(\beta_1 + \frac{\alpha_1}{\pi} - \frac{1}{n})\right] + \frac{\gamma_2^2}{n^2\pi^2} + \frac{\pi^2\left(\frac{Cn}{a}\right)^2}{8(1-\sigma)}(1+4\gamma_2^2)^2\, 2(1+\frac{\sigma}{\pi^2})(\sigma+\frac{1}{\pi^2})\right.$$

$$\left. - \frac{\hat{\sigma}}{E}(1+\sigma)(\sigma+\frac{1}{\pi^2})(1+4\gamma_2^2)\right\} - 1.5\left(\frac{Cn}{a}\right)\gamma_2\left[(\pi+\frac{1}{\pi})(\alpha_1 + \frac{\beta_1}{\pi}) + \frac{3}{4n}(\sigma+\frac{1}{\pi^2})\right]$$

$$+ \left(\frac{Cn^2}{a}\right)\left\{ (\pi+\frac{1}{\pi})^2(2.5\gamma_2^4 + 2\gamma_2^2 + \frac{5}{32}) + (\pi-\frac{1}{\pi})^2(2\gamma_2^4 + \gamma_2^2 + \frac{1}{8})\right\}$$

$$+ \frac{\left(\frac{tn}{a}\right)^2}{12}\left\{ (\pi+\frac{1}{\pi})^2(1 + 16\gamma_2^2)\right\} = 0$$

$$n^2\left(\alpha_1 + \frac{\beta_1}{\pi} + \frac{1}{n\pi}\right)^2 = \left[\frac{\left(\frac{Cn^2}{a}\right)\gamma_2\,(\pi+\frac{1}{\pi})(\pi^2-1+\frac{1}{\pi^2})}{2(\pi^2-1+\frac{1}{\pi^2}) + \frac{3}{4}(1-\sigma)} - \frac{\frac{2}{\pi}(\pi^2-1+\frac{1}{\pi^2}) - \frac{1-\sigma}{2}(\pi - \frac{1}{2\pi})}{2(\pi^2-1+\frac{1}{\pi^2}) + \frac{3}{4}(1-\sigma)} + \frac{1}{\pi}\right]^2$$

$$= \left(1.680786\left(\frac{Cn^2}{a}\right)\gamma_2 + 0.065578\right)^2$$

$$n^2\beta_1^2 = \left[0.020450 + 0.234663\left(\frac{Cn^2}{a}\right)\gamma_2\right]^2$$

$$\frac{n\alpha_1}{\pi} = \left[-0.0783763 + 0.558763\left(\frac{Cn^2}{a}\right)\gamma_2\right]$$

$$n\left(\beta_1+\frac{\alpha_1}{\pi}-\frac{1}{n}\right)=\frac{-\frac{2}{\pi^2}-\frac{1-\sigma}{4\pi}+2\left(\frac{1}{\pi}-1\right)+\frac{1-\sigma}{2}+\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\gamma_2\left[\pi-\frac{1}{2\pi}-\frac{\pi}{2}+\frac{1}{\pi}\right]}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}-1 \qquad 87)$$

$$=\frac{-\left[2-\frac{1-\sigma}{4}\right]+\left(\frac{Cn}{a}\right)\left(\pi+\frac{1}{\pi}\right)\frac{1}{2}\left(\pi+\frac{1}{\pi}\right)\gamma_2}{2\left(\pi^2-1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}-1$$

$$=-1.098826+0.324120\left(\frac{Cn^2}{a}\right)\gamma_2$$

$$\frac{2\pi^2\left(1+\frac{\sigma}{\pi^2}\right)\left(\sigma+\frac{1}{\pi^2}\right)}{8(1-\sigma)}=\frac{11.66076}{8}=\cancel{1.457595}\ 1.45760$$

$$(1+\sigma)\left(\sigma+\frac{1}{\pi^2}\right)=0.521716$$

$$\left[1.68077\left(\frac{Cn^2}{a}\right)\gamma_2+0.065578\right]^2+0.35\Big\{\left[0.020450+0.234663\left(\frac{Cn^2}{a}\right)\gamma_2\right]^2$$

$$+\left[-0.0783763+0.558783\left(\frac{Cn^2}{a}\right)\gamma_2\right]\left[-1.098826+0.324120\left(\frac{Cn^2}{a}\right)\gamma_2\right]\Big\}$$

$$+\frac{\gamma_2^2}{\pi^2}+1.45760\left(\frac{Cn^2}{a}\right)^2\left(1+4\gamma_2^2\right)^2-0.521716\left(\frac{\sigma n^2}{E}\right)\left(1+4\gamma_2^2\right)$$

$$-8.72298\left(\frac{Cn^2}{a}\right)^2\gamma_2^2+0.86016\left(\frac{Cn^2}{a}\right)\gamma_2$$

$$+\left(\frac{Cn}{a}\right)^2\Big\{11.97094\left(2.5\gamma_2^4+2\gamma_2^2+\frac{5}{32}\right)+7.97099\left(2\gamma_2^4+\gamma_2^2+\frac{1}{8}\right)\Big\}$$

$$+0.997578\left(\frac{tn^2}{a}\right)^2\left(1+16\gamma_2^2\right)=0$$

88)

$$69.1909\left(\frac{Cn^2}{a}\right)^2\gamma_2^4+\left\{37.7583\left(\frac{Cn^2}{a}\right)^2+0.10132-2.08686\left(\frac{\sigma n^2}{E}\right)+15.9612\left(\frac{tn^2}{a}\right)^2\right\}\gamma_2^2$$

$$+\gamma_2\left(\frac{Cn^2}{a}\right)0.86016+\left\{0.997578\left(\frac{tn^2}{a}\right)^2-0.521716\left(\frac{\sigma n^2}{E}\right)+0.0345895+4.32443\left(\frac{Cn^2}{a}\right)^2\right\}=0.$$

$$\boxed{\begin{aligned}&69.1909\,x^2\gamma_2^2+37.7583\,x^2+\left[0.10132-2.08686\left(\frac{\sigma n^2}{E}\right)+15.9612\left(\frac{tn^2}{a}\right)^2\right]\gamma_2^2\\&+0.86016\,x+4.32443\,\frac{x^2}{\gamma_2^2}+\left[0.997578\left(\frac{tn^2}{a}\right)^2-0.521716\left(\frac{\sigma n^2}{E}\right)+0.0345895\right]=0.\end{aligned}}$$

$\cancel{0.69191\,\xi^2\gamma_2^2+}$

$$\boxed{\begin{aligned}&\left\{0.69191\,\xi^2+0.10132-2.08686\left(\frac{\sigma n^2}{E}\right)+0.159612\,\eta^2\right\}\gamma_2^2+\frac{0.0432443\,\xi^2}{\gamma_2^2}\\&+\left\{0.377583\,\xi^2+0.086016\,\xi+0.0099758\,\eta^2-0.521716\left(\frac{\sigma n^2}{E}\right)+0.0345895\right\}=0\end{aligned}}$$

Corrected form

$$\left\{0.69191\,\xi^2+0.10132-2.08686\left(\frac{\sigma n^2}{E}\right)+0.159612\,\eta^2\right\}\gamma_2^2+\frac{0.0432443\,\xi^2}{\gamma_2^2}$$

$$+\left\{0.377583\,\xi^2+0.086016\,\xi+\left[0.0099758+\frac{0.00125110}{n^4}(0.020450+0.023466\,\xi)^2\right.\right.$$

$$\left.\left.-\frac{1}{n^2}\,0.0030022\,(0.020450+0.023466\,\xi)\right]\eta^2-0.521716\left(\frac{\sigma n^2}{E}\right)+0.0345895\right\}=0.$$

$\xi=\left(\frac{Cn^2}{a}\right)\gamma_2 \quad 10$

$C =\quad \gamma$

The neglected term:

$$\left(2-2\sigma+\frac{1}{\pi^2}\right)\left(\frac{\beta_1}{n}\right)^2+2\left(2+\frac{1}{\pi^2}-\sigma\right)\left(\frac{\beta_1}{n}\right)$$

$$\frac{\beta_1}{n} = -\frac{1}{n^2}\;\frac{\frac{2}{\pi^2}+\frac{1-\sigma}{4}+\left(\frac{Cn^2}{a}\right)\left(\pi+\frac{1}{\pi}\right)\left(\frac{\pi}{2}-\frac{1}{\pi}\right)\gamma_2}{2\left(\pi^2 1+\frac{1}{\pi^2}\right)+\frac{3}{4}(1-\sigma)}$$

$$= -\frac{1}{n^2}\left(0.020450 + 0.23466\left(\frac{Cn^2}{a}\right)\gamma_2\right)$$

$$= -\frac{1}{n^2}\left[0.020450 + 0.23466\,x\right]$$

$$\left(2-2\sigma+\frac{1}{\pi^2}\right)\left(\frac{\beta_1}{n}\right)^2 = \frac{1}{n^4}\;1.50132\left(0.020450+0.23466\,x\right)^2$$

$$+\;-2\left(2+\frac{1}{\pi^2}-\sigma\right)\frac{\beta_1}{n} = -\frac{1}{n^2}\;3.60264\left(0.020450+0.23466\,x\right)$$

$$\cancel{\frac{1}{n^2}\left(0.020450+0.23466x\right)\left[3.60264+\frac{x\cdot 3070}{n^2}+\frac{0.35220\,x}{n^2}\right]}$$

$$\cancel{+\frac{1}{n^4}}$$

The additional terms will be

$$\left(\frac{Cn^2}{a}\right)^2\left[\frac{1}{n^4}\;0.125110\left(0.020450+0.23466x\right)^2-\frac{1}{n^2}\;0.30022\left(0.020450+0.23466x\right)\right]$$

$$- \frac{\xi(0.057344 + 0.435748)[0.691915\xi^2 + 0.10132 - 2.08686\phi + 0.159612\eta^2]}{1.38382\xi^2 + 0.20264 + 0.31923\eta^2 - 4.17373\phi}$$

$$- \frac{0.0432443\xi[1.38382\xi^2 + 0.20264 + 0.31923\eta^2 - 4.17373\phi]}{(0.057344 + 0.43574\xi)}$$

$$+ (0.377583\xi^2 + 0.086016\xi + 0.0099758\eta^2 - 0.521716\phi + 0.0345895) = 0$$

$$- \xi\Big\{(0.057344 + 0.43574\xi)^2[0.69191\xi^2 + 0.10132 - \underline{2.08686\phi} + 0.159612\eta^2]$$

$$+ 0.043244[1.38382\xi^2 + 0.20264 + 0.31923\eta^2 - \underline{4.173734\phi}]^2\Big\}$$

$$+ (0.057344 + 0.43574\xi)(1.38382\xi^2 + 0.20264 + 0.31923\eta^2 - \underline{4.173734\phi})$$
$$(0.37758\xi^2 + 0.086016\xi + 0.0099758\eta^2 - \underline{0.521716\phi} + 0.0345895) = 0$$

When $\eta = 1.000$

$$- \xi\Big\{(0.057344 + 0.43574\xi)^2(0.691915\xi^2 + 0.26093 - \underline{2.08686\phi})$$

$$+ 0.043244(1.383825\xi^2 + 0.52187 - \underline{4.173734\phi})^2\Big\}$$

$$+ (0.057344 + 0.43574\xi)(1.38382\xi^2 + 0.52187 - \underline{4.173734\phi})(0.37758\xi^2 + 0.086016\xi$$

$$+ 0.044566 - \underline{0.521716\phi}) = 0.$$

91)

$$A\phi^2 + B\phi + C = 0$$

$$A = 4.17373 \times 0.521716\,(0.057344 + 0.43574\xi) - 4.17373^2 \times 0.043244\,\xi$$

$$= 4.17373\left[0.029917 + 0.227333\xi - 0.180493\xi\right]$$

$$\boxed{A = 4.17373\,(0.029917 + 0.046843\xi)}$$

$$B = -\xi\left\{-2.08686\,(0.057344 + 0.43574\xi)^2 - 2\times 0.043244 \overset{4.17373}{} (1.38382\xi^2 + 0.52187)\right\}$$

$$-\ (0.057344 + 0.43574\xi)\left[0.521716\,(1.38382\xi^2 + 0.52187) + 4.17373\,(0.37758\xi^2 + 0.086016\xi + 0.044566)\right]$$

$$= \xi\left\{2.08686\,(0.057344 + 0.43574\xi)^2 + 2\times 0.043244 \times 4.17373\,(1.38382\xi^2 + 0.52187)\right\}$$

$$-\ (0.057344 + 0.43574\xi)(2.29788\xi^2 + 0.35901\xi + 0.45827)$$

$$B = \xi\left\{2.08686\,(0.057344 + 0.43574\xi)^2 + 0.36098\,(1.38382\xi^2 + 0.52187)\right\}$$

$$-\ (0.057344 + 0.43574\xi)(2.29788\xi^2 + 0.35901\xi + 0.45827)$$

$$C = -\xi\left\{(0.057344 + 0.43574\xi)^2(0.69191\xi^2 + 0.26093) + 0.043244\,(1.38382\xi^2 + 0.52187)^2\right\}$$

$$+\ (0.057344 + 0.43574\xi)(1.38382\xi^2 + 0.52187)(0.37758\xi^2 + 0.086016\xi + 0.044566)$$

p. 27

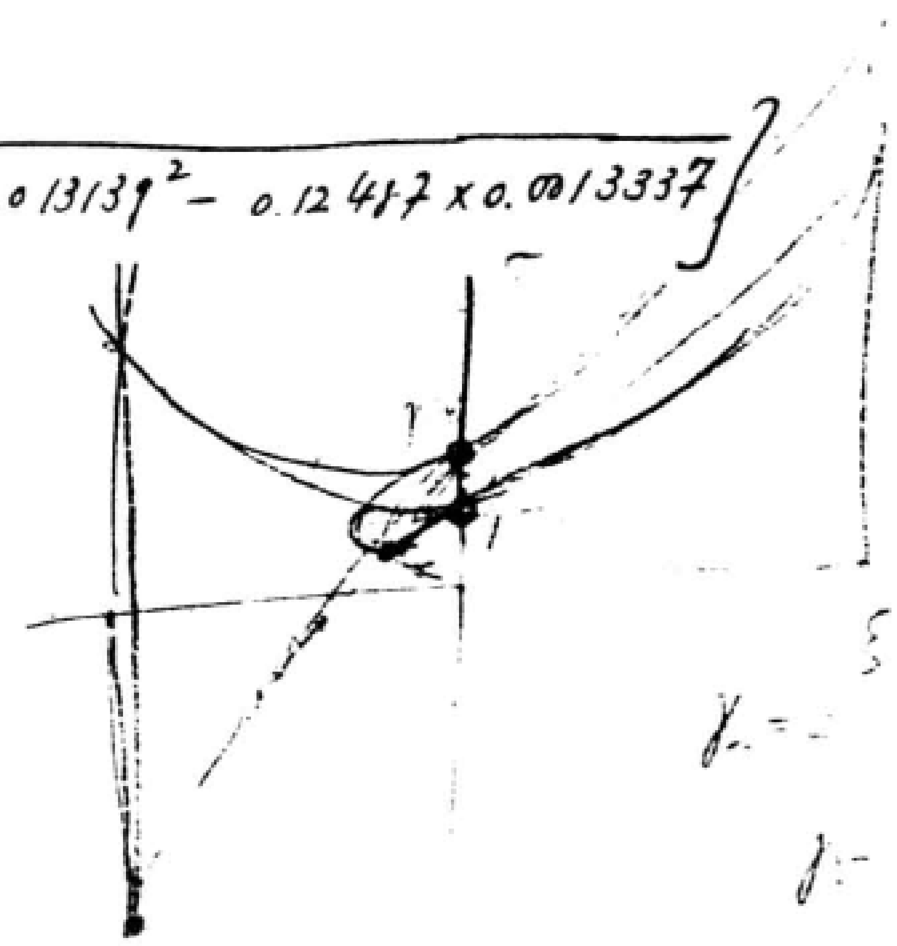
for $\xi = 0$

$A = 0.12487$ $\quad B = -0.02679$

$C = +0.0013337$

$$\phi = \frac{1}{0.12487}\left[0.013139 \pm \sqrt{0.013139^2 - 0.12487 \times 0.0013337}\right]$$

$$= \begin{matrix} +0.0855 \\ +0.1250 \end{matrix}$$



for $\xi = 1$

$A = 0.32036$

$B = -0.34074$

$C = 0.08880$

$$\phi = \frac{1}{0.32036}\left[0.17037 \pm \sqrt{0.17037^2 - 0.32036 \times 0.08880}\right]$$

$$= \frac{1}{0.32036}\left[0.17037 \pm \sqrt{0.029026 - 0.028448}\right] = \left.\begin{matrix} +0.457 \\ +0.607 \end{matrix}\right\}$$

for $\xi = -1$

$A = -0.07063$

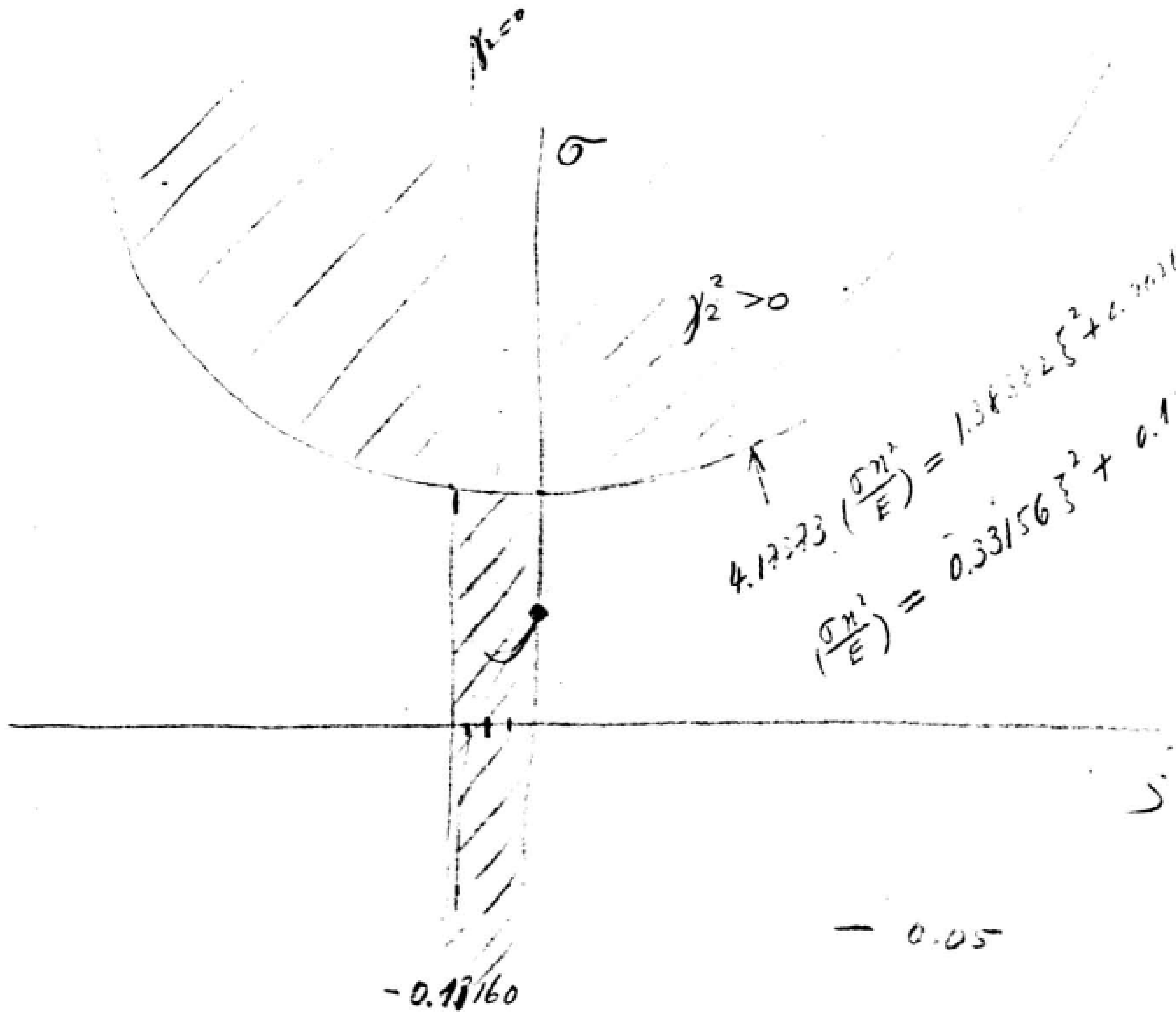
$B = -(0.29882 + 0.68792) + 0.37840 \times 2.39714$

$= -0.98674 + 0.90708 = -0.07966$

$C = +0.14319 \times 0.95284 + 0.15705 - 0.37840 \times 1.90569 \times 0.33613$

$= 0.13644 + \ldots - \ldots = 0.5110$

$$\phi = -\frac{1}{0.7063}\left[0.3483 \pm \sqrt{(0.3983)^2 + 0.706 \times 0.511}\right] = \begin{matrix} +0.455 \\ -1.543 \end{matrix}$$



$$4.17373\left(\frac{\sigma\pi^2}{E}\right) = 1.38382\,\xi^2 + 0.\ldots + 0.31473$$

$$\left(\frac{\sigma\pi^2}{E}\right) = 0.33156\,\xi^2 + 0.12504$$

0.01
0.0033

0.01
0.0033156
0.1250
0.1283

0.0036

0.021194
0.1250
1262

Let $\xi = -0.13160$

$$A = 4.17373\,(0.029917 - 0.006164) = 0.099138$$

$$B = -0.13160 \times 0.36098 \times 0.54584 = -0.025930$$

$$C = +0.13160 \times 0.043244 \times 0.54584^2 = +0.0016956$$

$$\frac{\sigma n^2}{E} = \phi = \frac{1}{0.99138}\left[0.12965 \pm \sqrt{0.12965^2 - 0.99138 \times 0.016956}\right]$$

$$= \frac{1}{0.99138}\left[0.12965 \pm 0\right] =$$

$$\xi = -0.100$$

$$A = 4.17373\,(0.029917 - 0.004684) = 0.10532$$

$$B = -0.100\,(2.08686 \times 0.013770^2 + 0.36098 \times 0.53571)$$

$$- 0.013770 \times 0.44535 = -0.100\,(2.08686 \times 0.00018961 + 0.36098 \times 0.53571)$$

$$- 0.0061325$$

$$= -0.100\,(0.0003957 + 0.19338) - 0.0061325$$

$$= -0.019378 - 0.006133 = -0.025511$$

$$C = 0.100\,(0.013770^2 \times 0.26785 + 0.043244 \times 0.53571^2)$$

$$+ 0.013770 \times 0.53571 \times 0.039740$$

$$= 0.0012461 + 0.0002932 = 0.0015393$$

$$\phi = \frac{1}{1.0532}\left[0.12756 \pm \sqrt{0.12756^2 - 1.0532 \times 0.015393}\right]$$

$$= \frac{1}{1.0532}\left[0.12756 \pm 0.00775\right] = \begin{matrix} 0.1137 \\ 0.1284 \end{matrix}$$

$$\xi = -0.050$$

$$A = 4.17323\,(0.029917 - 0.002342) = 0.11509$$

$$B = -0.050\,(2.04686 \times 0.035557^2 + 0.36096 \times 0.52533)$$

$$- 0.035557 \times 0.44606$$

$$= -0.050\,(2.04686 \times 0.0012643 + 0.36096 \times 0.52533) - 0.015861$$

$$= -0.050 \times 0.19226 - 0.015861 = -0.025474$$

$$C = 0.050\,(0.0012643 \times 0.26266 + 0.043244 \times 0.52533^2)$$

$$+ 0.035557 \times 0.52533 \times 0.041209$$

$$= 0.050 \times 0.012266 + 0.0007697 = 0.0013830$$

$$\phi = \frac{1}{1.1509}\left[+0.12737 \pm \sqrt{0.12737^2 - 1.1509 \times 0.013830} \right]$$

$$= \frac{1}{1.1509}\left[0.12737 \pm \sqrt{0.000306} \right]$$

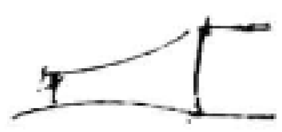
$$= \begin{matrix} 0.0954 \\ 0.1259 \end{matrix}$$

1000.00

00

700.

$$\frac{F_x}{F_{max}} \sim \frac{f_x}{f_{c.\text{ф}}}$$



When $\xi = -0.050$

$$0 = +0.050\Big[0.0012643(0.10305 + 0.159612\eta^2 - 2.08686\phi) +$$

$$+0.043244(0.20610 + 0.31923\eta^2 - 4.17373\phi)^2\Big]$$

$$+\ 0.035557(0.20610 + 0.31923\eta^2 - 4.1737\phi)(0.031233 + 0.0099758\eta^2 - 0.521716\phi)$$

$$\phi^2\big[+0.050 \times 0.043244 \times 4.17373^2 + 0.03557 \times 4.17373 \times 0.521716\big]$$

$$= \phi^2\big[+0.037666 + 0.077454\big] = 0.1151\phi^2$$

Coefficient of ϕ

$$+0.050\big[-0.0012643 \times 2.08686 - 2 \times 0.043244 \times 4.17373(0.20610 + 0.31923\eta^2)\big]$$

$$-\ 0.035557\big[4.17373(0.031233 + 0.0099758\eta^2) + 0.521716(0.20610 + 0.31923\eta^2)\big] -$$

$$= -\Big[0.050(0.0026384 + 0.074398 + 0.11524\eta^2)$$

$$+\ 0.035557(0.23789 + 0.20819\eta^2)\Big]$$

$$\boxed{= -(0.0123105 + 0.013165\eta^2)}$$

Constant term

$$+0.050\big[0.0012643(0.10305 + 0.159612\eta^2) + 0.043244(0.20610 + 0.31923\eta^2)^2\big]$$

$$+0.035557(0.20610 + 0.31923\eta^2)(0.031233 + 0.0099758\eta^2)$$

$$= \Big[0.00063215(0.10305 + 0.159612\eta^2) + 0.021622(0.20610 + 0.31923\eta^2)^2$$

$$+\ 0.35557(0.20610 + 0.31923\eta^2)(0.031233 + 0.0099758\eta^2)\Big]10^{-1}$$

97)

$$= 10^{-1}\Big[0.0006514 + 0.0010090\eta^2 + 0.0009184 + 0.0024452\eta^2 + 0.0022035\eta^4$$

$$+ 0.0022888 + 0.0042763\eta^2 + 0.0011235\eta^4\Big]$$

$$= 10^{-1}\Big[0.0032723 + 0.0072224\eta^2 + 0.0033359\eta^4\Big]$$

$$1.151\phi^2 - (0.1231 + 0.1317\eta^2)\phi + (0.0032723 + 0.0072224\eta^2 + 0.0033359\eta^4) = 0.$$

$$\phi^2 - (0.1069 + 0.1144\eta^2)\phi + (0.002840 + 0.00627\eta^2 + 0.002900\eta^4) = 0$$

$$\left(\frac{\sigma n^2}{E}\right)^2 - \left[0.1069 + 11.44\left(\frac{tn^2}{a}\right)^2\right]\left(\frac{\sigma n^2}{E}\right)$$

$$+ \left[0.002840 + 0.627\left(\frac{tn^2}{a}\right)^2 + 29.00\left(\frac{tn^2}{a}\right)^4\right] = 0.$$

$$\left(\frac{\sigma}{E}\right)^2 - \frac{\sigma}{E}\left[\frac{0.1069}{n^2} + 11.44n^2\left(\frac{t}{a}\right)^2\right] + \left[\frac{0.002840}{n^4} + 0.627\left(\frac{t}{a}\right)^2 + 29.00n^4\left(\frac{t}{a}\right)^4\right] = 0$$

or $$\psi^2 - \psi\left(\frac{0.1069}{n^2} + 11.44n^2\varsigma^2\right) + \left(\frac{0.002840}{n^4} + 0.627\varsigma^2 + 29.00n^4\varsigma^4\right) = 0.$$

$$-\psi\left(-\frac{0.1069}{n^4} + 11.44\varsigma^2\right) + \left(-\frac{0.005680}{n^6} + 58.00n^2\varsigma^4\right) = 0.$$

$$\psi = \frac{58.00n^2\varsigma^4 - \dfrac{0.005680}{n^6}}{11.44\varsigma^2 - \dfrac{0.1069}{n^4}}$$

$$= \frac{58.00n^6\varsigma^4 - \dfrac{0.005680}{n^2}}{11.44n^4\varsigma^2 - 0.1069}$$

98)

$$\frac{\left(58.00\, n^6 \zeta^4 - \dfrac{0.005680}{n^2}\right)^2}{(11.44\, n^4 \zeta^2 - 0.1069)^2} - \frac{\left(58.00\, n^6 \zeta^4 - \dfrac{0.005680}{n^2}\right)\left(\dfrac{0.1069}{n^2} + 11.44 n^2 \zeta^2\right)}{11.44\, n^4 \zeta^2 - 0.1069}$$

$$+ \left(\frac{0.002840}{n^4} + 0.627\zeta^2 + 29.00\, n^4 \zeta^4\right) = 0.$$

$$n^4 \left(58.00\, n^4 \zeta^4 - \frac{0.005680}{n^4}\right)^2 - \left(58.00\, n^8 \zeta^4 - 0.005680\right)\left(130.9\, n^4 \zeta^4 - \frac{0.01143}{n^4}\right)$$

$$+ (11.44\, n^4 \zeta^2 - 0.1069)^2 \left(\frac{0.002840}{n^4} + 0.627\zeta^2 + 29.00\, n^4 \zeta^4\right) = 0.$$

Let $n^4 = m$

$$m\left(58.00\, m \zeta^4 - \frac{0.005680}{m}\right)^2 - (58.00\, m^2 \zeta^4 - 0.005680)\left(130.9\, m\zeta^4 - \frac{0.01143}{m}\right)$$

$$+ (11.44\, m\zeta^2 - 0.1069)^2 \left(\frac{0.002840}{m} + 0.627\zeta^2 + 29.00\, m\zeta^4\right) = 0.$$

Put $\frac{t}{a} = \zeta = 10^{-3}$, $m = 10^4 \delta$

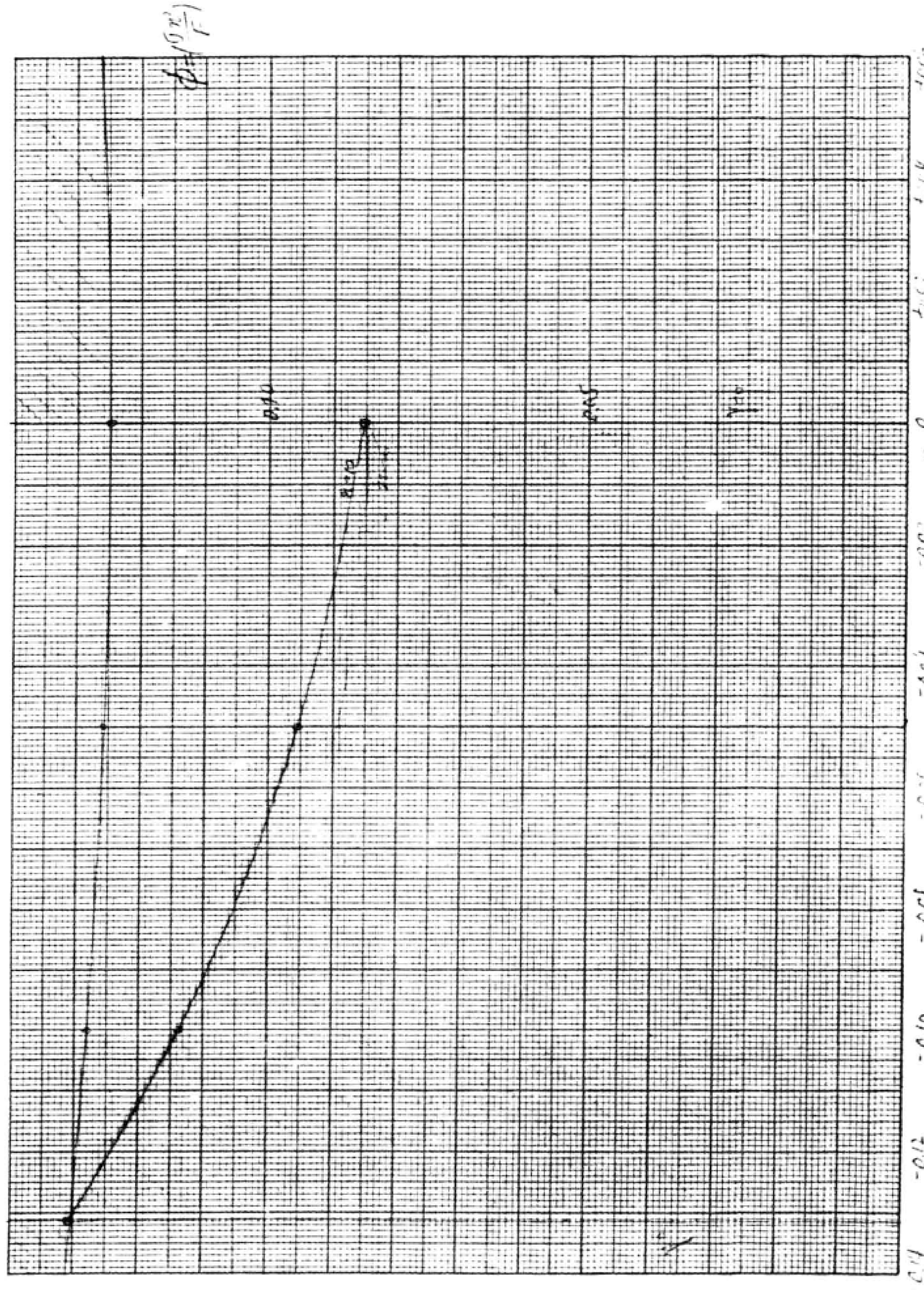
$$m\zeta^4 = \delta\, 10^{-8} \qquad m\zeta^2 = 10^{-2}\delta$$

$$m^2\zeta^4 = \delta^2 10^{-4}$$

$$\delta\left(0.0000005800\delta - \frac{0.0000005680}{\delta}\right)^2 - (0.0058\delta^2 - 0.005680)\left(0.000001309\delta - \frac{0.000001143}{\delta}\right)$$

$$+ (0.1144\delta - 0.1069)^2\left(\frac{0.0000002840}{\delta} + 0.000000627 + 0.0000002900\delta\right) = 0.$$

$\frac{\ell}{a} = \frac{1}{1000}$; $n = 10$;



$$\delta\left(0.5800\delta - \frac{0.5680}{\delta}\right)^2 - (0.58\delta^2 - 0.5680)\left(1.309\delta - \frac{1.143}{\delta}\right)$$

$$+ (1.144\delta - 1.069)^2\left(\frac{0.2240}{\delta} + 0.622 + 0.3900\delta\right) = 0.$$

If $\delta = 1.000$

$$0.0120^2 - 0.0120 \times 0.166 + 0.075^2 \times 1.201$$

$$= 0.000144 - 0.001992 + 0.006755 = \underline{\underline{+0.00491}}$$

If $\delta = 0.800$

$$0.800 \times (-0.2460)^2 - 0.800 \times 0.2460 \times 0.3816 + (-0.154)^2\, 1.214$$

$$= 0.04842 - 0.07510 + 0.02880 = +0.00212$$

If $\delta = 0.700$ $\qquad \therefore \qquad \delta \approx 0.700$

$$0.700(-0.4054)^2 - 0.700 \times 0.4054 \times 0.7166$$

$$+ (-0.268)^2 \times 1.236 = 0.11501 - 0.20336 + 0.08822$$

$$= 0$$

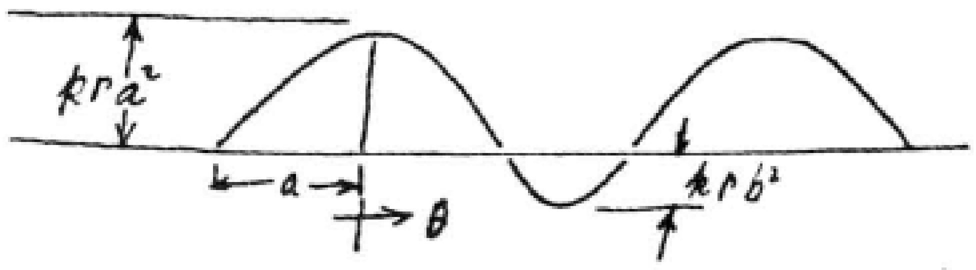
$m = n^4 = 7000 \qquad n^2 = 83.6$

$n^6 = 586000$

$$\psi = \frac{58.00 \times 586000 \times 10^{-12} - 0.00006800}{11.44 \times 7000 \times 10^{-6} - 0.1069}$$

$$= \frac{0.000034}{0.0267} = \frac{0.0034}{2.67} = 0.00127$$

Deflection in Parabolic Profile



Let the upper half of the wave be of the shape

$$kr\{a^2-(\theta)^2\}$$

where a is a pure number.

When $\theta = a$, the upper curve is ended.

Now the slope of the upper curve at $\theta = a$ is

$$\underline{\underline{-2kra}}$$

The ~~slope of~~ shape of the lower half of the wave can be expressed as

$$-hr[b^2-[\theta-(a+b)]^2]$$

The tangent at $\theta = a$ is $\quad hr[+2[a-(a+b)]] = -2hrb$

Equating the slope $\quad ka = hb$

$$\therefore \quad h = k\frac{a}{b}$$

Thus the shape of deflection,

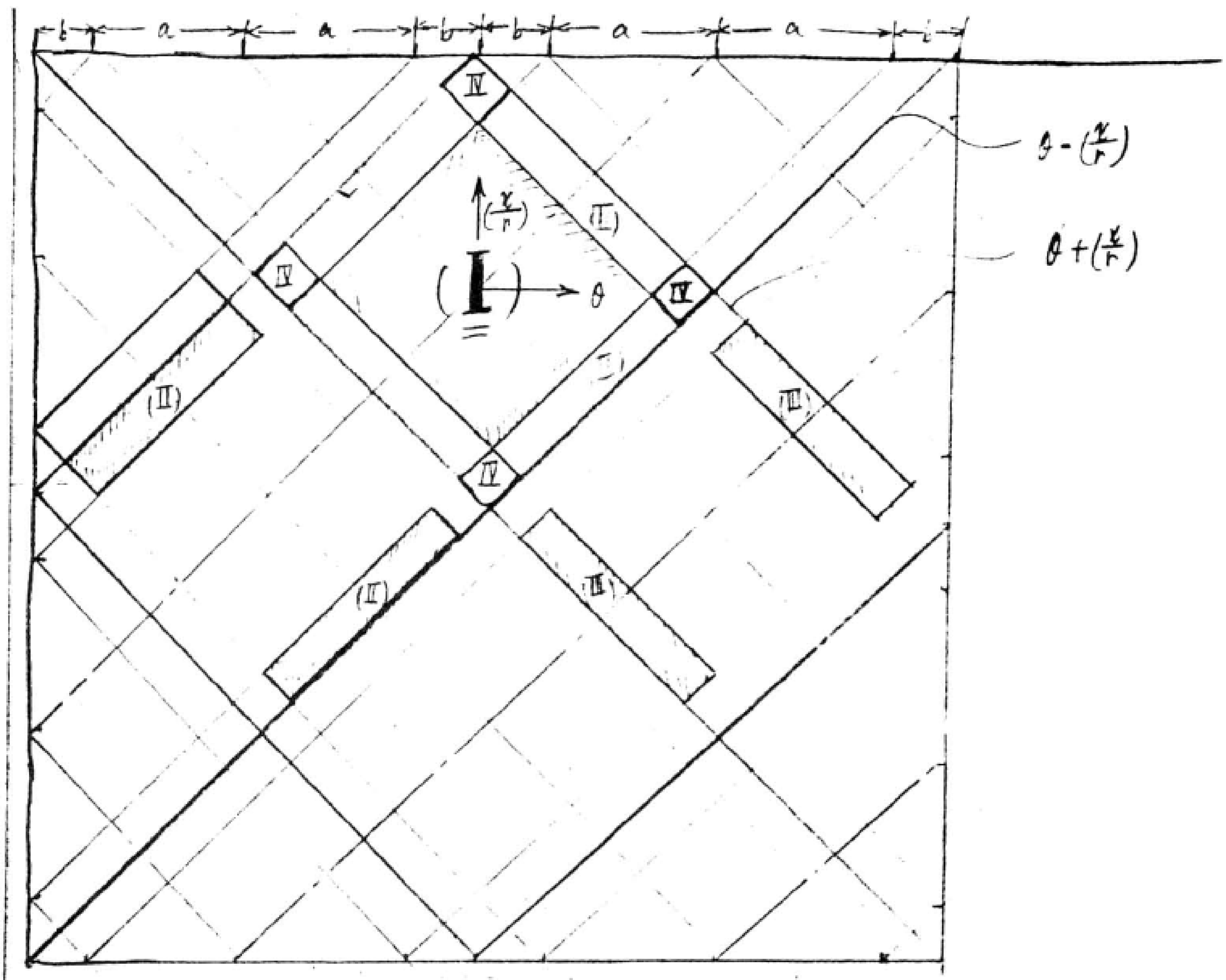
$$\begin{cases} w = kr\{a^2-\theta^2\} & \text{from } -a \le \theta \le +a \\ w = -k\frac{a}{b}r\{b^2-[\theta-(a+b)]^2\} & \text{for } +a \le \theta \le a+b \end{cases}$$

The wave length $= 2(a+b)$

Thus the final shape of the diamond wave will be: 707)

$$\cancel{w = \frac{kr}{2}\left\{a^2 - \left(\theta + \frac{x}{r}\right)^2 + a^2 - \left(\theta - \frac{x}{r}\right)^2\right\}}$$ O.K.

$$\cancel{= kr\left\{a^2 - \left[\theta^2 + \left(\frac{x}{r}\right)^2\right]\right\}}$$ ~~for region I~~,



~~for region (II)~~

$$\cancel{w = \frac{kr}{2}\left\{a^2 - \left(\theta + \frac{x}{r}\right)^2 - \frac{a}{b}\left[b^2 - \left\{\theta - \frac{x}{r} - (a+b)\right\}^2\right]\right\}}$$ O.K.

for region (III)

$$w = \frac{kr}{2}\left\{a^2 - \left(\theta - \frac{x}{r}\right)^2 - \frac{a}{b}\left[b^2 - \left\{\theta + \frac{x}{r} - (a+b)\right\}^2\right]\right\}$$ O.K.

~~For region (IV)~~

$$\cancel{w = -k\frac{a}{b}r\left[b^2 - \left\{\theta^2 + \left(\frac{x}{r}\right)^2 - 2(a+b)\theta + (a+b)^2\right\}\right]}$$ O.K

$$\varepsilon_1 = u_0' + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\varepsilon_2 = -\frac{w}{r} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$\gamma = \frac{1}{r}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial \theta}\right)$$

(I): $\frac{\partial w}{\partial x} = -2kr\,\frac{x}{r^2}$ $\qquad \varepsilon_1 = u_0 + 2k^2\left(\frac{x}{r}\right)^2$

$\frac{\partial w}{\partial \theta} = -2kr\,\theta$ $\qquad \varepsilon_2 = -k\left\{a^2 - \left[\theta^2 + \left(\frac{x}{r}\right)^2\right]\right\} + 2k^2\theta^2$

$$\gamma = 4k^2\theta\left(\frac{x}{r}\right)$$

$$\varepsilon_1 + \varepsilon_2 = u_0 + 2k^2\left[\left(\frac{x}{r}\right)^2 + \theta^2\right] + k\left\{\left[\theta^2 + \left(\frac{x}{r}\right)^2\right] - a^2\right\}$$

$$\gamma^2 - 4\varepsilon_1\varepsilon_2 = \underline{16k^4\theta^2\left(\frac{x}{r}\right)^2 - 16k^4\theta^2\left(\frac{x}{r}\right)^2}$$

$$-8u_0k^2\theta^2 + 8k^3\left(\frac{x}{r}\right)^2\left\{a^2 - \left[\theta^2 + \left(\frac{x}{r}\right)^2\right]\right\} + 4u_0k\left\{a^2 - \left[\theta^2 + \left(\frac{x}{r}\right)^2\right]\right\}$$

$$\gamma^2 - 4\varepsilon_1\varepsilon_2 = 8k^2\left\{k\left(\frac{x}{r}\right)^2\left[a^2 - \left(\theta^2 + \frac{x^2}{r^2}\right)\right] - u_0\theta^2\right\} + 4u_0k\left\{a^2 - \left[\theta^2 + \left(\frac{x}{r}\right)^2\right]\right\}$$

$$\underline{\varepsilon_1 + \varepsilon_2 = u_0 - ka^2 + \left[\theta^2 + \left(\frac{x}{r}\right)^2\right]\left\{2k^2 + k\right\}}$$

$\underline{k_1 = \frac{\partial^2 w}{\partial x^2} = -\frac{2k}{r}}$ $\qquad \underline{k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = -\frac{2k}{r}}$

$\underline{\tau = 0}$

$$(\varepsilon_1+\varepsilon_2)^2 = (u_0-ka^2)^2 + 2(2k^2+k)(u_0-ka^2)\left\{\theta^2+\left(\tfrac{x}{r}\right)^2\right\}$$
$$+ (2k^2+k)^2\left\{\theta^2+\left(\tfrac{x}{r}\right)^2\right\}^2$$

Make a coördinate transformation

$$s = \frac{1}{\sqrt{2}}\theta - \frac{1}{\sqrt{2}}\left(\frac{x}{r}\right) \qquad \text{or} \qquad \theta = \frac{1}{\sqrt{2}}s + \frac{1}{\sqrt{2}}t$$
$$t = \frac{1}{\sqrt{2}}\theta + \frac{1}{\sqrt{2}}\left(\frac{x}{r}\right) \qquad \qquad \frac{x}{r} = -\frac{1}{\sqrt{2}}s + \frac{1}{\sqrt{2}}t$$

$$\theta^2+\left(\frac{x}{r}\right)^2 = s^2+t^2$$

$$\underline{\iint(\varepsilon_1+\varepsilon_2)^2\,d\theta\,d\left(\tfrac{x}{r}\right)} = 4\int_0^{\frac{a}{\sqrt{2}}}\int_0^{\frac{a}{\sqrt{2}}}\left[(u_0-ka^2)^2 + 2(2k^2+k)(u_0-ka^2)(s^2+t^2) + (2k^2+k)^2(s^2+t^2)^2\right]ds\,dt$$

$$= 4\left\{(u_0-ka^2)^2\frac{a^2}{2} + \frac{2}{3}(2k^2+k)(u_0-ka^2)\frac{a^4}{2} + (2k^2+k)^2\left[\frac{1}{5}\frac{a^5}{4\sqrt{2}}\frac{a}{\sqrt{2}} + 2\frac{1}{9}\frac{a^6}{8}\right.\right.$$
$$\left.\left. + \frac{1}{5}\frac{a^5}{4\sqrt{2}}\frac{a}{\sqrt{2}}\right]\right\}$$

$$= 4\left\{(u_0-ka^2)\frac{a^2}{2} + \frac{1}{3}(2k^2+k)(u_0-ka^2)a^4 + (2k^2+k)^2\left[\frac{1}{20}+\frac{1}{36}\right]a^6\right\}$$

$$\underline{= 4a^2\left\{\frac{(u_0-ka^2)^2}{2} + \frac{(2k^2+k)(u_0-ka^2)a^2}{3} + \frac{7(2k^2+k)^2a^4}{90}\right\}} \;\; ////$$

$$\iint(\theta^2-4\varepsilon_1\varepsilon_2)\,d\theta\,d\left(\tfrac{x}{r}\right) = 4\int_0^{\frac{a}{\sqrt{2}}}\int_0^{\frac{a}{\sqrt{2}}} 8k^2\left\{\frac{k}{2}(s^2+2st+t^2)\left[a^2-(s^2+t^2)\right] - \frac{u_0}{2}(s^2-2st+t^2)\right\}ds\,dt$$

$$= 32k^2\int_0^{\frac{a}{\sqrt{2}}}\int_0^{\frac{a}{\sqrt{2}}}\left[\frac{ka^2}{2}(s^2+2st+t^2) - \frac{u_0}{2}(s^2-2st+t^2) - \frac{k}{2}(s^4+2s^3t+2s^2t^2+2st^3+t^4)\right]ds\,dt$$

$$= 32k^2\left[\frac{ka^2}{2}\left(\frac{1}{3}\frac{a^3}{2\sqrt{2}}\frac{a}{\sqrt{2}} + \frac{1}{2}\frac{a^4}{4} + \frac{1}{3}\frac{a^4}{4}\right) - \frac{u_0}{2}\left(\frac{2}{3}\frac{a^4}{4} - \frac{1}{2}\frac{a^4}{4}\right)\right.$$

$$- \frac{k}{2}\left(\frac{1}{5}\frac{a^5}{4\sqrt{2}}\frac{a}{\sqrt{2}} + 2\frac{1}{4}\frac{a^4}{4}\frac{1}{2}\frac{a^2}{2} + 2\frac{1}{3}\frac{a^3}{2\sqrt{2}}\frac{1}{3}\frac{a^3}{2\sqrt{2}} + 2\frac{1}{2}\frac{a^2}{2}\frac{1}{4}\frac{a^4}{4} + \frac{1}{5}\frac{a^5}{4\sqrt{2}}\frac{a}{\sqrt{2}}\right)$$

$$+ 16u_0k\left\{a^2\frac{a^2}{2} - \left[\frac{1}{3}\frac{a^3}{2\sqrt{2}}\frac{a}{\sqrt{2}}\;2\right]\right\}$$

$$\iint (\Delta^2 - 4\epsilon_1\epsilon_2)\, d\theta\, d\left(\frac{x}{r}\right) = 32k^2\left[\frac{7}{24}\,\frac{ka^6}{2} - \frac{u_0}{2}\,\frac{1}{24}\,a^4 - \frac{k}{2}a^6\,\frac{157}{360\times 4}\right]$$

$$= 32k^2\left[\frac{ka^6\cdot 263}{8\times 360} - \frac{u_0 a^4}{48}\right]$$

$$= \frac{2}{3}k^2a^4\left[\frac{263}{60}ka^2 - u_0\right] + \frac{16}{3}u_0 k a^4$$

~~see page (103)~~

$$\cancel{\iint (\epsilon_1+\epsilon_2)^2\, d\theta\, d\left(\frac{x}{r}\right) = 4a^2\left\{u_0\left(\frac{1}{2} - \frac{2k^2+k}{3}a^2\right) - \frac{ka^2}{2} + (2k^2+k)a^4\left[\frac{7}{45}k^2 - \frac{33}{90}k\right]\right\}}$$

$$(k_1+k_2)^2 - 2(1-\sigma)(k_1k_2 - \tau^2) = \frac{16k^2}{r^2} - 2(1-\sigma)\frac{4k^2}{r^2}$$

$$= \frac{8k^2}{r^2}\left[2 - (1-\sigma)\right] = \frac{8k^2}{r^2}(1+\sigma) \qquad \iint(\quad)\, d\theta\, d\left(\frac{x}{r}\right) = \frac{8k^2}{r^2}(1+\sigma)\frac{a^2}{2}4$$

For Region (II)

$$\frac{\partial w}{\partial x} = \frac{kr}{2}\left\{-2\frac{x}{r^2} - \frac{a}{b}\left[-2\left\{\theta - \frac{x}{r} - (a+b)\right\}\left(-\frac{1}{r}\right)\right]\right\}$$

$$= -k\left\{\left(\frac{x}{r}\right) + \frac{a}{b}\left[\theta - \left(\frac{x}{r}\right) - (a+b)\right]\right\}$$

$$= -k\left\{\frac{a}{b}\theta + \left(1-\frac{a}{b}\right)\left(\frac{x}{r}\right) - \frac{a}{b}(a+b)\right\}$$

$$\frac{\partial w}{\partial \theta} = \frac{kr}{2}\left\{-\right.$$

for region III) 105)

$$\frac{\partial w}{\partial x} = \frac{kr}{2}\left\{-2\left(\theta+\frac{x}{r}\right)\frac{1}{r} - \frac{a}{b}\left[-2\left\{\theta-\frac{x}{r}-(a+b)\right\}\left(-\frac{1}{r}\right)\right]\right\}$$

$$= -k\left\{\left(\theta+\frac{x}{r}\right) + \frac{a}{b}\left[\theta-\frac{x}{r}-(a+b)\right]\right\}$$

$$= -k\left\{\left(1+\frac{a}{b}\right)\theta + \left(1-\frac{a}{b}\right)\left(\frac{x}{r}\right) - \frac{a}{b}(a+b)\right\}$$

$$\frac{\partial w}{\partial \theta} = \frac{kr}{2}\left\{-2\left(\theta+\frac{x}{r}\right) - \frac{a}{b}\left[-2\left\{\theta-\frac{x}{r}-(a+b)\right\}\right]\right\}$$

$$= -kr\left\{\left(\theta+\frac{x}{r}\right) - \frac{a}{b}\left(\theta-\frac{x}{r}-(a+b)\right)\right\}$$

$$= -kr\left\{\left(1-\frac{a}{b}\right)\theta + \left(1+\frac{a}{b}\right)\left(\frac{x}{r}\right) + \frac{a}{b}(a+b)\right\}$$

$$\varepsilon_1 = u_0 + \frac{k^2}{2}\left\{\left(1+\frac{a}{b}\right)\theta + \left(1-\frac{a}{b}\right)\left(\frac{x}{r}\right) - \frac{a}{b}(a+b)\right\}^2$$

$$\varepsilon_2 = -\frac{k}{2}\left\{a^2-\left(\theta+\frac{x}{r}\right)^2 - \frac{a}{b}\left[b^2-\left\{\theta-\frac{x}{r}-(a+b)\right\}^2\right]\right\} + \frac{k^2}{2}\left\{\left(1-\frac{a}{b}\right)\theta+\left(1+\frac{a}{b}\right)\frac{x}{r} + \frac{a}{b}(a+b)\right\}^2$$

$$A = k^2\left\{\left(1+\frac{a}{b}\right)\theta + \left(1-\frac{a}{b}\right)\left(\frac{x}{r}\right) - \frac{a}{b}(a+b)\right\}\left\{\left(1-\frac{a}{b}\right)\theta + \left(1+\frac{a}{b}\right)\left(\frac{x}{r}\right) + \frac{a}{b}(a+b)\right\}$$

$$\frac{\partial^2 w}{\partial x^2} = -\frac{k}{r}\left(1-\frac{a}{b}\right) \qquad k_1 = -\frac{k}{r}\left(1-\frac{a}{b}\right)$$

$$\frac{\partial^2 w}{\partial \theta^2} = -kr\left(1-\frac{a}{b}\right) \qquad k_2 = -\frac{k}{r}\left(1-\frac{a}{b}\right)$$

$$\frac{\partial^2 w}{\partial x \partial \theta} = -k\left(1+\frac{a}{b}\right) \qquad \tau = -\frac{k}{r}\left(1+\frac{a}{b}\right)$$

Now put $\quad \xi = \frac{1}{\sqrt{2}} s + \frac{1}{\sqrt{2}} t$ 106)

$$\frac{\eta}{r} = -\frac{1}{\sqrt{2}} s + \frac{1}{\sqrt{2}} t$$

$$\varepsilon_1 = u_0 + \frac{k^2}{2}\left\{\left(1+\frac{a}{b}\right)\left(\frac{1}{\sqrt{2}} s + \frac{1}{\sqrt{2}} t\right) + \left(1-\frac{a}{b}\right)\left(-\frac{1}{\sqrt{2}} s + \frac{1}{\sqrt{2}} t\right) - \frac{a}{b}(a+b)\right\}^2$$

$$= u_0 + \frac{k^2}{2}\left\{\frac{a}{b}\sqrt{2}\, s + \sqrt{2}\, t - \frac{a}{b}(a+b)\right\}^2$$

$$= u_0 + \frac{k^2}{2}\left\{\sqrt{2}\left(\frac{a}{b} s + t\right) - \frac{a}{b}(a+b)\right\}^2$$

$$\varepsilon_2 = -\frac{k}{2}\left\{a^2 - 2t^2 - \frac{a}{b}\left[b^2 - \left\{\sqrt{2}\, s - (a+b)\right\}^2\right]\right\}$$

$$+ \frac{k^2}{2}\left\{\sqrt{2}\left(-\frac{a}{b} s + t\right) + \frac{a}{b}(a+b)\right\}^2$$

$$\varpi = k^2\left\{\sqrt{2}\left(\frac{a}{b} s + t\right) - \frac{a}{b}(a+b)\right\}\left\{\sqrt{2}\left(-\frac{a}{b} s + t\right) + \frac{a}{b}(a+b)\right\}$$

Now put $\quad s = p + \frac{a}{\sqrt{2}}$

$$\varepsilon_1 = u_0 + \frac{k^2}{2}\left\{\sqrt{2}\left(\frac{a}{b} p + t\right) - a\right\}^2$$

$$\varepsilon_2 = -\frac{k}{2}\left\{-a^2 - 2t^2 - \frac{a}{b}\left[b^2 - \left\{\sqrt{2}\, p - b\right\}^2\right]\right\}$$

$$+ \frac{k^2}{2}\left\{\sqrt{2}\left(-\frac{a}{b} p + t\right) + a\right\}^2$$

$$\varpi = k^2\left\{\sqrt{2}\left(\frac{a}{b} p + t\right) - a\right\}\left\{\sqrt{2}\left(-\frac{a}{b} p + t\right) + a\right\}$$

107)

$$(\mathcal{E}_1+\mathcal{E}_2)^2 = \left[u_0 + \frac{k^2}{2}\left\{\sqrt{2}\left(\frac{a}{b}p+t\right)-a\right\}^2\right]^2$$

$$+ \frac{k^2}{4}\left[a^2-2t^2-\frac{a}{b}\left\{b^2-(\sqrt{2}p-b)^2\right\} - k\left\{\sqrt{2}\left(-\frac{a}{b}p+t\right)+a\right\}^2\right]^2$$

$$- \frac{k}{2}\left[u_0+\frac{k^2}{2}\left\{\sqrt{2}\left(\frac{a}{b}p+t\right)-a\right\}^2\right]\left[a^2-2t^2-\frac{a}{b}\left\{b^2-(\sqrt{2}p-b)^2\right\}-k\left\{\sqrt{2}\left(-\frac{a}{b}p+t\right)+a\right\}^2\right]$$

$$A^2-4\mathcal{E}_1\mathcal{E}_2 = 2u_0k\left\{a^2-2t^2-\frac{a}{b}\left[b^2-(\sqrt{2}p-b)^2\right]\right\} - 2u_0k^2\left\{\sqrt{2}\left(-\frac{a}{b}p+t\right)+a\right\}^2$$

$$+ k^3\left\{\sqrt{2}\left(\frac{a}{b}p+t\right)-a\right\}^2\left\{a^2-2t^2-\frac{a}{b}\left[b^2-(\sqrt{2}p-b)^2\right]\right\}$$

Integrating $8\overline{\displaystyle\int_{p=0}^{\frac{b}{\sqrt{2}}}\int_{t=0}^{\frac{a}{\sqrt{2}}}(\mathcal{E}_1+\mathcal{E}_2)^2\,ds\,dt}$

$$\int_{p=0}^{\frac{b}{\sqrt{2}}}\int_{t=0}^{\frac{a}{\sqrt{2}}}\left[u_0+\frac{k^2}{2}\left\{\sqrt{2}\left(\frac{a}{b}p+t\right)-a\right\}^2\right]^2 dp\,dt$$

$$=\iint\left[\left(u_0+\frac{k^2a^2}{2}\right)-\sqrt{2}k^2a\left(\frac{a}{b}p+t\right)+k^2\left(\frac{a}{b}p+t\right)^2\right]^2 dp\,dt$$

$$=\iint\Big[\left(u_0+\frac{k^2a^2}{2}\right)^2+2k^4a^2\left(\frac{a}{b}p+t\right)^2+k^4\left(\frac{a}{b}p+t\right)^4$$

$$-2\sqrt{2}k^2\left(u_0+\frac{k^2a^2}{2}\right)a\left(\frac{a}{b}p+t\right)+2k^2\left(u_0+\frac{k^2a^2}{2}\right)\left(\frac{a}{b}p+t\right)^2$$

$$-2\sqrt{2}k^4a\left(\frac{a}{b}p+t\right)^3\Big]dp\,dt$$

$$=\iint\Big[\left(u_0+\frac{k^2a^2}{2}\right)^2+k^2(3k^2a^2+2u_0)\left(\frac{a}{b}p+t\right)^2-2\sqrt{2}k^2a\left(u_0+\frac{k^2a^2}{2}\right)\left(\frac{a}{b}p+t\right)$$

$$-2\sqrt{2}k^4a\left(\frac{a}{b}p+t\right)^3+k^4\left(\frac{a}{b}p+t\right)\Big]dp\,dt$$

$$= \left(u_0 + \frac{k^2a^2}{2}\right)^2 \frac{ab}{2} - k^2 a^3 b \left(u_0 + \frac{k^2a^2}{2}\right) + \frac{7}{24} a^3 b \left(3k^2a^2 + 2u_0\right) k^2$$

$$- \frac{3}{4} k^4 b a^5 + \frac{31}{120} b a^5 k^4$$

$$= \left(u_0^2 + u_0 k^2 a^2\right) \frac{ab}{2} - k^2 a^3 b u_0 + \frac{7}{12} a^3 b u_0 k^2$$

$$+ a^5 b \left[\frac{k^4}{8} - \frac{k^4}{2} + \frac{7}{8} k^4 - \frac{3}{4} k^4 + \frac{31}{120} k^4 \right]$$

$$= \frac{ab}{2} u_0^2 + \frac{1}{12} a^3 b u_0 k^2 + \frac{1}{120} k^4 a^5 b$$

$$\left[(a^2 - 2t^2) - \frac{a}{b}\left\{b^2 - (\sqrt{2}p - b)^2\right\} - k\left\{\sqrt{2}\left(-\frac{a}{b}p + t\right) + a\right\}^2\right]^2$$

$$= (a^2 - 2t^2)^2 + \frac{a^2}{b^2}\left\{b^2 - (\sqrt{2}p - b)^2\right\}^2 + k^2\left\{\sqrt{2}\left(-\frac{a}{b}p + t\right) + a\right\}^4$$

$$- 2\left(\frac{a}{b}\right)(a^2 - 2t^2)\left\{b^2 - (\sqrt{2}p - b)^2\right\} - 2k(a^2 - 2t^2)\left\{\sqrt{2}\left(-\frac{a}{b}p + t\right) + a\right\}^2$$

$$+ 2k\left(\frac{a}{b}\right)\left\{b^2 - (\sqrt{2}p - b)^2\right\}\left\{\sqrt{2}\left(-\frac{a}{b}p + t\right) + a\right\}^2$$

$$= (a^2 - 2t^2)^2 + \frac{a^2}{b^2}\left\{b^4 - 2b^2(\sqrt{2}p - b)^2 + (\sqrt{2}p - b)^4\right\}$$

$$+ k^2\left\{4\left(-\frac{a}{b}p + t\right)^4 + 4 \times 2\sqrt{2}\, a\left(-\frac{a}{b}p + t\right)^3 + 6 \times 2\, a^2\left(-\frac{a}{b}p + t\right)^2 + 4\sqrt{2}\, a^3\left(-\frac{a}{b}p + t\right)\right.$$

$$\left. + a^4 \right\}$$

$$- 2\left(\frac{a}{b}\right)(a^2 - 2t^2)\left\{b^2 - (\sqrt{2}p - b)^2\right\} - 2k(a^2 - 2t^2)\left\{2\left(-\frac{a}{b}p + t\right)^2 + 2\sqrt{2}\, a\left(-\frac{a}{b}p + t\right) + a^2\right\}$$

$$+ 2k\left(\frac{a}{b}\right)\left\{b^2 - (\sqrt{2}p - b)^2\right\}\left\{2\left(-\frac{a}{b}p + t\right)^2 + 2\sqrt{2}\, a\left(-\frac{a}{b}p + t\right) + a^2\right\}$$

$\iint (\qquad) \, dp\, dt$

$\frac{1}{2} - \frac{1}{3} + \frac{1}{10}$

$\frac{15-10+3}{30}$

$$= a^4 \frac{ab}{2} - \cancel{4}a^2 \frac{b}{\sqrt{2}} \frac{1}{3} \frac{a^3}{2\sqrt{2}} + \cancel{4} \frac{b}{\sqrt{2}} \frac{1}{5} \frac{a^5}{4\sqrt{2}}$$

$$+ \frac{a^2}{b^2} \left\{ b^4 \frac{ab}{2} - 2b^2 \frac{ab^3}{6} + \frac{ab^5}{10} \right\}$$

$$+ k^2 \left\{ \frac{4}{120} a^5 b + a^5 b + \qquad \right\} -$$

$$- 2\left(\frac{a}{b}\right) a^2 \left\{ \frac{ab^3}{2} - \frac{ab^3}{6} \right\} + 4\left(\frac{a}{b}\right) \frac{1}{18} a^3 b^3$$

$$- 2ka^2 \left\{ 2 \cdot \frac{1}{24} a^3 b + \frac{a^3 b}{2} \right\} + 4k \left\{ \frac{11}{720} a^5 b + \frac{a^5 b}{4} + a^2 \frac{\ }{2} \frac{a^3}{2\sqrt{2}} \frac{b}{\sqrt{2}} \right\}$$

$$+ 4k\left(\frac{a}{b}\right) \left\{ \frac{19 a^3 b^3}{720} - \frac{a^3 b^3}{24} + \frac{a^3 b^3}{6} \right\}.$$

$$= \frac{4}{15} a^5 b + \frac{4 a^3 b^3}{15} + \frac{31}{30} k^2 a^5 b - \frac{4 a^4 b^2}{9} - \frac{109}{180} k a^5 b$$

$$+ k a^4 b^2 \frac{109}{180}$$

$$= \frac{a^3}{15} \left[\left(4 - \frac{109}{12} k + \frac{31}{2} k^2\right) a^2 b + \left(\frac{109}{12} k - \frac{20}{3} \right) a b^2 + 4 b^3 \right]$$

to be multiplied by $\frac{k^2}{4}$

$$\left[\left(u_0+\frac{k^2a^2}{2}\right)-\sqrt{2}\,k^2a\left(\frac{a}{b}p+t\right)+k^2\left(\frac{a}{b}p+t\right)^2\right] \quad 110)$$

$$\left[(a^2-2t^2)-2\frac{a}{b}\left(\sqrt{2}\,bp-p^2\right)-k\left\{2\left(-\frac{a}{b}p+t\right)^2+2\sqrt{2}\,a\left(-\frac{a}{b}p+t\right)+a^2\right\}\right]$$

$$=\left[(a^2-2t^2)-2\frac{a}{b}\left(\sqrt{2}\,bp-p^2\right)\right]\left[\left(u_0+\frac{k^2a^2}{2}\right)-\sqrt{2}\,k^2a\left(\frac{a}{b}p+t\right)+k^2\left(\frac{a}{b}p+t\right)^2\right]$$

$$-k\left[\left(u_0+\frac{k^2a^2}{2}\right)-\sqrt{2}\,k^2a\left(\frac{a}{b}p+t\right)+k^2\left(\frac{a}{b}p+t\right)^2\right]\left[2\left(-\frac{a}{b}p+t\right)^2+2\sqrt{2}\,a\left(-\frac{a}{b}p+t\right)+a^2\right]$$

$$\overset{\int}{=}\quad a^2\left(u_0+\frac{k^2a^2}{2}\right)\frac{ab}{2}-\sqrt{2}\,k^2a^3\frac{a^2b}{2\sqrt{2}}+k^2a^2\frac{7}{24}a^3b$$

$$-2\left(u_0+\frac{k^2a^2}{2}\right)\frac{1}{3}\frac{a^3}{\sqrt{2}}\frac{b}{\sqrt{2}}+2\sqrt{2}\,k^2a\frac{a^4b}{2\sqrt{2}}\frac{5}{24}-2k^2\frac{50.5}{720}a^5b$$

$$-2\left(\frac{a}{b}\right)\left[\left(u_0+\frac{k^2a^2}{2}\right)\frac{ab^3}{6}-\sqrt{2}\,k^2a\frac{a^2b^3}{2\sqrt{2}}\frac{3}{8}+k^2\frac{84.5}{720}a^3b^3\right]$$

$$-k\left(u_0+\frac{k^2a^2}{2}\right)\left[\frac{7}{12}a^3b\right]$$

$$+\sqrt{2}\,k^3a\left\{\frac{a^4b}{\sqrt{2}\cdot 12}+\frac{a^3b}{2\sqrt{2}}\right\}-k^3\left\{\frac{2a^5b}{45}+\frac{7}{24}a^5b\right\}$$

$$=\left\{\frac{a^3b}{2}-\frac{1}{6}ab^3-\frac{a^2b^2}{3}-\frac{7a^3bk}{12}\right\}u_0+k^2a^5b\left[\frac{1}{4}-\frac{1}{2}+\frac{7}{24}-\frac{1}{6}+\frac{5}{24}-\frac{101}{720}\right]$$

$$+k^3a^5b\left[-\frac{7}{24}+\frac{1}{12}+\frac{1}{2}-\frac{2}{45}-\frac{7}{24}\right]$$

$$-a^4b^2k^2\left[\frac{1}{6}-\frac{3}{8}+\frac{169}{720}\right]$$

$$=ab\left[\frac{a^2}{2}-\frac{ab}{3}-\frac{b^2}{6}-\frac{7a^2k}{12}\right]u_0-a^5bk^2\frac{41}{720}-\frac{2a^5bk^3}{45}-a^4b^2k^2\frac{19}{720}$$

$$=ab\left[\frac{a^2}{2}-\frac{ab}{3}-\frac{b^2}{6}-\frac{7a^2k}{12}\right]u_0-\frac{41}{720}a^5bk^2-\frac{2}{45}a^5bk^3-\frac{19}{720}a^4b^2k^2$$

multiply by $-\frac{k}{2}$

not in following exp.

$\boxed{8}\iint \mathscr{H}^2 - 4\epsilon_1\epsilon_2 = 2u_0 k\left[\frac{a^3b}{2} - \frac{8}{3}\frac{a^3}{2\sqrt{2}}\frac{b}{\sqrt{2}} - 2\left(\frac{a}{b}\right)\frac{ab^3}{6}\right]$ 111)

$-2u_0k^2\left\{2\cdot\frac{a^3b}{24} + \frac{a^3b}{2}\right\}$ $\frac{1}{12}+\frac{1}{2}$ $\frac{7}{12}$

$+k^3a^2\left\{2\cdot\frac{7}{24}a^3b - 2\sqrt{2}a\frac{a^2b}{2\sqrt{2}} + \frac{a^3b}{2}\right\}$

$-2k^3\left\{2\,\frac{50.5}{720}a^5b - 2\sqrt{2}a\frac{a^4b}{2\sqrt{2}}\frac{5}{24} + \frac{1}{3}\frac{a^4}{2\sqrt{2}}\frac{b}{\sqrt{2}}\right\}$

$-2\left(\frac{a}{b}\right)k^3\left\{\frac{169}{720}a^3b^3 - 2\sqrt{2}a\frac{a^2b^3}{2\sqrt{2}}\frac{3}{8} + \frac{a^3b^3}{6}\right\}$

$= 2u_0k\left[\frac{1}{3}a^3b - \frac{1}{3}a^2b^2\right] - \frac{7}{6}u_0k^2a^3b$ $3\frac{-11}{360}$

$+\frac{k^3a^5b}{12} - \frac{22k^3a^5b}{720} - 2a^4b^2k^3\times\frac{19}{720}$

$= \frac{2}{3}u_0k\left(a^3b - a^2b^2\right) - \frac{7}{6}u_0k^2a^3b + \frac{19}{360}k^3a^5b - \frac{19}{360}k^3a^4b^2$

$~~(k_1+k_2)^2 = \frac{4k^2}{r^2}\left(1-\frac{a}{b}\right)^2 \ldots = \frac{4k^2}{r^2}\left(1-\frac{a}{b}\right)^2\frac{a}{2}~~$

$(k_1+k_2)^2 - 2(1-\sigma)(k_1k_2-\tau^2)$

$= \frac{4k^2}{r^2}\left(1-\frac{a}{b}\right)^2 - 2(1-\sigma)\left[\frac{k^2}{r^2}\left(1-\frac{a}{b}\right)^2 - \frac{k^2}{r^2}\left(1+\frac{a}{b}\right)^2\right]$

$= \frac{k^2}{r^2}\left\{4\left(1-\frac{a}{b}\right)^2 - 2(1-\sigma)\left(-4\frac{a}{b}\right)\right\}$

$= 4\frac{k^2}{r^2}\left\{\left(1-\frac{a}{b}\right)^2 + 2(1-\sigma)\left(\frac{a}{b}\right)\right\}$

$\iint(\quad)\,ds\,dt = 8\times 4\frac{k^2}{r^2}\frac{ab}{2}\left\{\left(1-\frac{a}{b}\right)^2 + 2(1-\sigma)\frac{a}{b}\right\}$

For region IV, the strain energy must be of same form as for region I, only $k \longrightarrow -k\frac{a}{b}$ $a \longrightarrow b$

Then $\iint (\varepsilon_1+\varepsilon_2)^2 d\theta\, d\frac{z}{r} = 4b^2\left\{\frac{(u_0+k\,a\,b)^2}{2}+\frac{(2k^2a^2-kab)(u_0+kab)}{3}\right.$

$\left.+\frac{7}{90}(2k^2a^2-kab)^2\right\}$

$\iint (\omega^2-4\varepsilon_1\varepsilon_2)\,d\theta\, d\left(\frac{z}{r}\right) = \frac{2}{3}k^2\left(\frac{a}{b}\right)^2 b^4\left[-\frac{263}{60}k\left(\frac{a}{b}\right)b^2-u_0\right]-\frac{16}{3}u_0(k)\left(\frac{a}{b}\right)b^4$

The bending energy factor $= \frac{8k^2\left(\frac{a}{b}\right)^2}{r^2}(1+\sigma)\frac{b^2}{2}4$.

The total extensional energy for one diamond wave is

$$\frac{2W_e(1-\sigma^2)}{\left(\frac{t}{r}\right)E\cdot r^3} = 4a^2\left\{\frac{(u_0-ka^2)^2}{2}+\frac{(2k^2+k)(u_0-ka^2)a^2}{3}+\frac{7}{10}(2k^2+k)^2a^4\right\}$$

$$+\frac{1-\sigma}{2}\left\{\frac{2}{3}k^2a^4\left(\frac{263}{60}ka^2-u_0\right)+\frac{16}{3}u_0ka^4\right\}$$

$$+\left\{4a^2\xi u_0^2+\frac{2}{3}a^4\xi u_0k^2+\frac{1}{15}k^4a^6\xi\right\}+\frac{2k^2a^3}{15}\left\{\left(4-\frac{109}{12}k+\frac{31}{2}k^2\right)a^3\xi\right.$$

$$\left.+\left(\frac{109}{12}k-\frac{20}{3}\right)a^3\xi^2+4a\xi^3\right\}-4a^2\xi k\left\{\frac{a^2}{2}-\frac{a^2\xi}{3}-\frac{a^2\xi^2}{6}-\frac{7a^2k}{12}\right\}u_0$$

$$+\frac{41}{180}a^6\xi k^2+\frac{8}{45}a^6\xi k^4+\frac{19}{180}a^6\xi^2k^3$$

$$+\frac{1-\sigma}{2}\left\{\frac{16}{3}u_0k(a^4\xi-a^4\xi^2)-\frac{28}{3}u_0k^2a^4\xi+\frac{19}{45}k^3a^6\xi-\frac{19}{45}k^3a^6\xi^2\right\}$$

$$+4a^2\xi^2\left\{\frac{(u_0+ka^2\xi)^2}{2}+\frac{(2k^2a^2-ka^2\xi)(u_0+ka^2\xi)}{3}+\frac{7}{90}(2k^2a^2-ka^2\xi)^2\right\}$$

$$+\frac{1-\sigma}{2}\left\{\frac{2}{3}k^2a^4\xi^2\left[-\frac{263}{60}ka^2\xi-u_0\right]-\frac{16}{3}u_0k\,a^4\xi^3\right\}$$

The bending energy

$$\frac{2W_b\,(1-\sigma^2)}{\left(\frac{t}{r}\right)E\,r^3} = \frac{\left(\frac{t}{r}\right)^2}{12}\left[32k^2(1+\sigma)a^2 + 16k^2a^2\xi\left\{(1-\xi)^2 + 2(1-\sigma)\xi\right\}\right]$$

The number of waves in a circumference $= \frac{2\pi}{2(a+b)} = \frac{\pi}{a+b}$

$$= \frac{\pi}{a}\frac{1}{1+\xi}$$

For a cylinder of $2(a+b)r = 2a(1+\xi)r$ deep, we have

$\frac{2\pi}{a}\frac{1}{1+\xi}$ diamond waves.

The potential energy $= -\sigma\, u_0\, 2a(1+\xi)r \cdot 2\pi r \cdot t$

The potential energy per diamond wave

$$= -\sigma\, u_0\, 2a^2(1+\xi)^2 \cdot r^2 \cdot t$$

$$\frac{\text{Potential energy per diamond wave} \cdot 2(1-\sigma^2)}{\left(\frac{t}{r}\right)E \cdot r^3}$$

$$= -\sigma\, u_0\, 2a^2(1+\xi)^2 \cdot \cancel{r^2 t} \cdot \frac{2(1-\sigma^2)}{\cancel{\left(\frac{t}{r}\right)}E \cdot \cancel{r^3}}$$

$$= -4\left(\frac{\sigma}{E}\right) a^2(1+\xi)^2(1-\sigma^2)\,u_0$$

$$= -4\,\phi\, a^2(1+\xi)^2(1-\sigma^2)\,u_0$$

$\frac{\partial V}{\partial u_0} = 0$ gives 114)

$$0 = 4a^2\left\{ u_0 - ka^2 + \frac{a^2(2k^2+k)}{3}\right\} + \frac{1-\sigma}{2}\left\{ \frac{16}{3}ka^4 - \frac{2}{3}k^2a^4\right\}$$

$$+ \left\{8a^2\xi u_0 + \frac{2}{3}a^4\xi k^2\right\} - 4a^2\xi k\left\{\frac{a^2}{2} - \frac{a^2\xi}{3} - \frac{a^2\xi^2}{6} - \frac{7a^2k}{12}\right\}$$

$$+ \frac{1-\sigma}{2}\left\{\frac{16}{3}k\,(a^4\xi - a^4\xi^2) - \frac{28}{3}k^2a^4\xi\right\}$$

$$+ 4a^2\xi^2\left\{u_0 + ka^2\xi + \frac{(2k^2a^2 - ka^2\xi)}{3}\right\} - \frac{1-\sigma}{2}\left\{\frac{16}{3}ka^4\xi^3 + \frac{2}{3}k^2a^4\xi^2\right\}$$

$$- 4\phi\, a^2(1+\xi)^2(1-\sigma^2) = 0$$

$$u_0\left[4a^2 + 8a^2\xi + 4a^2\xi^2\right] = -4a^2\left\{\frac{a^2(2k^2+k)}{3} - ka^2\right\} - \frac{1-\sigma}{2}\left\{\frac{16}{3}ka^4 - \frac{2}{3}k^2a^4\right\}$$

$$- \frac{2}{3}a^4\xi k^2 + 4a^2\xi k\left\{\frac{1}{2} - \frac{\xi}{3} - \frac{\xi^2}{6} - \frac{7k}{12}\right\}a^2 - \frac{1-\sigma}{2}\left\{\frac{16}{3}k\,(\xi-\xi^2) - \frac{28}{3}k^2\xi\right\}a^4$$

$$- 4a^4\xi^2\left\{k\xi + \frac{(2k^2 - k\xi)}{3}\right\} + \frac{1-\sigma}{2}a^4\left\{\frac{16}{3}k\xi^3 + \frac{2}{3}k^2\xi^2\right\}$$

$$+ 4a^2(1+\xi)^2(1-\sigma^2)\phi$$

$$4u_0\,(1+\xi)^2 = 4(1+\xi)^2(1-\sigma^2)\phi - 4a^2\left\{\frac{2k^2+k}{3} - k\right\} - a^2\left(\frac{1-\sigma}{2}\right)\left\{\frac{16}{3}k - \frac{2}{3}k^2\right\}$$

$$- \frac{2}{3}a^2\xi k^2 + 4a^2\xi k\left\{\frac{1}{2} - \frac{\xi}{3} - \frac{\xi^2}{6} - \frac{7k}{12}\right\} - \left(\frac{1-\sigma}{2}\right)a^2\left\{\frac{16}{3}k\,(\xi-\xi^2) - \frac{28}{3}k^2\xi\right\}$$

$$- 4a^2\xi^2\left\{k\xi + \frac{(2k^2-\xi k)}{3}\right\} + \left(\frac{1-\sigma}{2}\right)a^2\left\{\frac{16}{3}k\xi^3 + \frac{2}{3}k^2\xi^2\right\}$$

Thus

$$u_0 = (1-\sigma^2)\phi + \frac{a^2}{4}\frac{1}{(1+\xi)^2}\left\{-\frac{8k(k-1)}{3} - \frac{2}{3}k^2\xi + 4\xi k\left(\frac{1}{2} - \frac{\xi}{3} - \frac{\xi^2}{6} - \frac{7k}{12}\right)\right.$$

$$\left. - \frac{8\xi^2k\,(k+\xi)}{3} + \left(\frac{1-\sigma}{2}\right)\left[-\frac{16}{3}k + \frac{2}{3}k^2 - \frac{16}{3}k\,(\xi-\xi^2) + \frac{28}{3}k^2\xi + \frac{16}{3}k\xi^3 + \frac{2}{3}k\xi^2\right]\right\}$$

115)

$$\frac{2W_e(1-\sigma^2)}{\left(\frac{t}{r}\right)r^3 E}\left(\frac{1}{\underline{\underline{a^2}}}\right) = 2(1+\xi)^2 u_0^2 + a^2\left\{\frac{4}{3}(2k^2+k) + 4k - \left(\frac{1-\sigma}{2}\right)\left[\frac{8}{3}k^2 - \frac{16}{3}k\right]\right.$$

$$+ \frac{2}{3}\xi k^2 - 4\xi k\left(\frac{1}{2} - \frac{\xi}{3} - \frac{\xi^2}{6} - \frac{7k}{12}\right) + \left(\frac{1-\sigma}{2}\right)\left[\frac{16}{3}k(\xi - \xi^2 k) - \frac{22}{3}k^2\xi\right]$$

$$\left. + \frac{4}{3}\xi^2(2k^2 - k\xi) + 4k\xi^3 - \left(\frac{1-\sigma}{2}\right)\left[\frac{2}{3}k^2\xi^2 + \frac{16}{3}k\xi^3\right]\right\} u_0$$

$$+ a^4\left\{2k^2 - \frac{4}{3}k(2k^2+k) + \frac{14}{45}(2k^2+k)^2 + \left(\frac{1-\sigma}{2}\right)\left[\frac{263}{90}k^3\right] + \frac{1}{15}k^4\xi\right.$$

$$+ \frac{2k^2}{15}\left[\left(4 - \frac{109}{12}k + \frac{31}{2}k^2\right)\xi + \left(\frac{109}{12}k - \frac{20}{3}\right)\xi^2 + 4\xi^3\right] + \frac{41}{180}\xi k^2$$

$$+ \frac{8}{45}\xi k^4 + \frac{19}{180}\xi^2 k^3 + \left(\frac{1-\sigma}{2}\right)\left[\qquad + \frac{19}{45}k^3\xi - \frac{19}{45}k^3\xi^2\right]$$

$$+ 2k^2\xi^4 + \frac{4}{3}(2k^2 - k\xi)k\xi^3 + \frac{14}{45}(2k^2 - k\xi)^2\xi^2$$

$$- \left(\frac{1-\sigma}{2}\right)\left[\frac{263}{90}k^3\xi^3\right]$$

$$= 2(1+\xi)^2 u_0^2 + a^2 u_0\left\{\frac{2k}{3}\left(\xi^3 + 2\xi^4 + \xi - 4\right) + k^2\left(3\xi + \frac{16}{3}\right)\right.$$

$$\left. + (1-\sigma)\left[\frac{8}{3}k\left(1 + \xi - \xi^3\right) - \frac{k^2}{3}\left(1 + 14\xi + 9\xi^2\right)\right]\right\}$$

$$+ a^4\left\{k^4\left(\frac{112}{45} + \frac{404}{45}\xi\right) + k^3\left[-\frac{64}{45} - \frac{19}{90}\xi + \frac{237}{180}\xi^2\right.\right.$$

$$\left. + (1-\sigma)\left(\frac{263}{180} + \frac{19}{90}\xi - \frac{19}{90}\xi^2 - \frac{263}{180}\xi^3\right)\right]$$

$$+ k^2\Big[$$

$$\frac{2M_e(1-\sigma^2)}{\left(\frac{t}{r}\right) r^3 E}\left(\frac{1}{a^2}\right) = 2(1+\xi)^2 u_0^2 + a^2 u_0 \Big\{ \frac{1}{3}k^2(8+9\xi+8\xi^2) - \frac{2}{3}k(4+3\xi-2\xi^2+5\xi^3)$$

$$+ (1-\sigma)\left[\frac{4}{3}k(1+\xi-\xi^2) - \frac{1}{3}k^2(1+14\xi+9\xi^2)\right]\Big\}$$

$$+ a^4 \Big\{ \frac{8}{45}k^4(7+13\xi+7\xi^2) + \frac{k^3}{45}\Big[\left(64\xi^3+\frac{237}{4}\xi^2-\frac{109}{2}\xi-64\right)$$

$$+ (1-\sigma)\left(\frac{263}{4}+\frac{19}{2}\xi-\frac{17}{2}\xi^2-\frac{263}{4}\xi^3\right)\Big]$$

$$+ \frac{k^2}{45}\left[44+\frac{105}{4}\xi-40\xi^2+24\xi^3+44\xi^4\right]\Big\} \tag{116}$$

$$\varepsilon_1 = u_0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \text{— (1)}$$

$$\varepsilon_2 = 0 = \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a} + \frac{1}{2a^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \text{— (2)}$$

$$\gamma = 0 = \frac{1}{a}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} \qquad \text{— (3)}$$

Differentiate (1) with respect to $\frac{1}{a}\frac{\partial}{\partial \theta}$ & (3) $\frac{\partial}{\partial x}$,

$$0 = \cancel{\frac{1}{a}\frac{\partial^2 u}{\partial x \partial \theta}} + \cancel{\frac{1}{a}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial \theta}}$$

$$0 = \cancel{\frac{1}{a}\frac{\partial^2 u}{\partial x \partial \theta}} + \frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial \theta} + \cancel{\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial \theta}}\right)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial \theta} = 0 \qquad \text{—— (4)}$$

Differentiate (4) with respect to $\frac{1}{a}\frac{\partial}{\partial \theta}$ and (2) $\frac{\partial^2}{\partial x^2}$,

$$\cancel{\frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \theta}} + \frac{1}{a^2}\left[\cancel{\frac{\partial^3 w}{\partial x^2 \partial \theta}\frac{\partial w}{\partial \theta}} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \theta^2}\right] = 0$$

$$\frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \theta} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2}\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial x \partial \theta}\right]$$

$$= \cancel{\frac{1}{a}\frac{\partial^3 v}{\partial x^2 \partial \theta}} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2}\left[\left(\frac{\partial^2 w}{\partial x \partial \theta}\right)^2 + \cancel{\frac{\partial w}{\partial \theta}\frac{\partial^3 w}{\partial x^2 \partial \theta}}\right]$$

$$\boxed{\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x \partial \theta}\right)^2 - \frac{1}{a^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2} = 0}$$

Or $\qquad \boxed{(s^2 - rt) - ar = 0}$

We see that the w-deflection for region I

$$w = kr\left[a^2 - \left(\theta^2 + \frac{x^2}{r^2}\right)\right]$$

satisfies the partial differential equation

$$-\frac{4k^2}{r^2} + \frac{2k}{r^2} = 0$$

or $\underline{k = \frac{1}{2}}$

$$\frac{\partial w}{\partial x} = -\left(\frac{x}{r}\right)$$

Thus $w = \frac{r}{2}\left[a^2 - \left(\theta^2 + \frac{x^2}{r^2}\right)\right]$ $\qquad \frac{1}{r}\frac{\partial w}{\partial \theta} = -\theta$

We have the relation $\frac{\partial u}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$

$$= -\frac{1}{2}\left(\frac{x}{r}\right)^2$$

$$\underline{\frac{u}{r} = f(\theta) - \frac{1}{6}\left(\frac{x}{r}\right)^3}$$

Also $\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{w}{r} - \frac{1}{2}\left(\frac{\partial w}{\partial \theta}\right)^2\frac{1}{r^2} = \frac{1}{2}\left\{a^2 - \left(\theta^2 + \frac{x^2}{r^2}\right)\right\} - \frac{1}{2}\theta^2$

$$= \frac{1}{2}a^2 - \theta^2 - \frac{1}{2}\frac{x^2}{r^2}$$

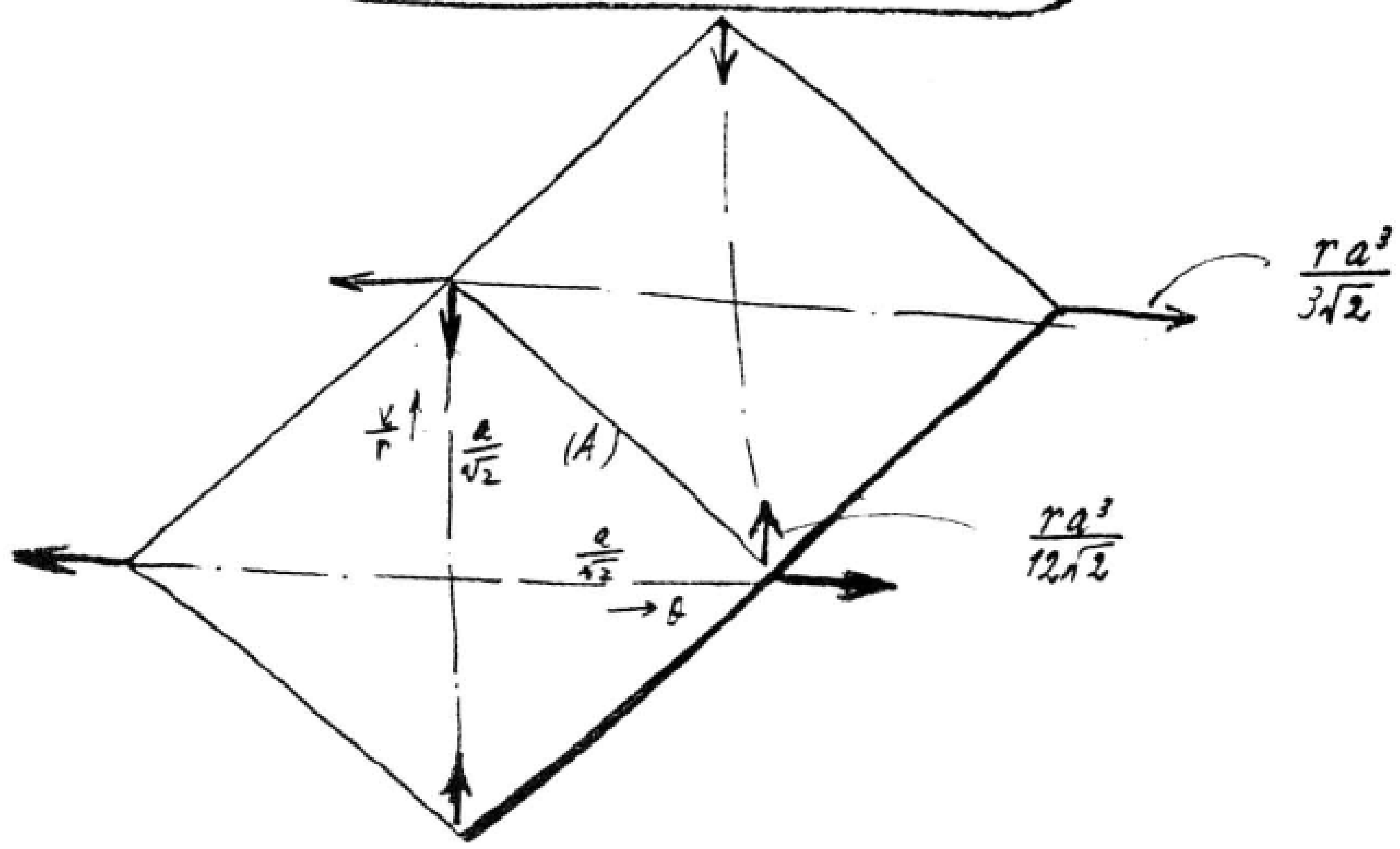
$$\underline{\frac{v}{r} = \frac{1}{2}a^2\theta - \frac{1}{3}\theta^3 - \frac{1}{2}\frac{x^2}{r^2}\theta + g\left(\frac{x}{r}\right)}$$

Now $-\frac{1}{r}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}$ or

$$-\theta\left(\frac{x}{r}\right) = f'(\theta) - \left(\frac{x}{r}\right)\theta + g'\left(\frac{x}{r}\right)$$

Thus $f'(\theta) = 0 = g'\left(\frac{x}{r}\right)$, $\therefore$ put $f(\theta) = 0 = g\left(\frac{x}{r}\right)$

Thus
$$\boxed{\begin{aligned} w &= \frac{p}{2}\left[a^2-\left(\theta^2+\frac{x^2}{r^2}\right)\right] \\ u &= -\frac{p}{6}\left(\frac{x}{r}\right)^3 \\ v &= \frac{p}{2}\left(a^2-\frac{x^2}{r^2}\right)\theta-\frac{r}{3}\theta^3 \end{aligned}}$$



Along the boundary (A), $\frac{a}{\sqrt{2}} = \theta + \left(\frac{x}{r}\right)$

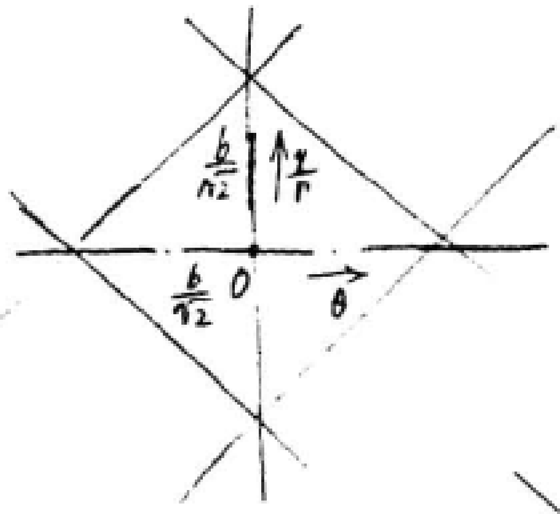
$$v = \frac{p}{2}\left(a^2-\frac{x^2}{r^2}\right)\left(\frac{a}{\sqrt{2}}-\frac{x}{r}\right)-\frac{r}{3}\left(\frac{a}{\sqrt{2}}-\frac{x}{r}\right)^3$$

$$\frac{v}{p} = \frac{1}{2}\left\{\frac{a^3}{\sqrt{2}}-a^2\left(\frac{x}{r}\right)-\frac{a}{\sqrt{2}}\left(\frac{x}{r}\right)^2+\left(\frac{x}{r}\right)^3\right\}$$

$$-\frac{1}{3}\left\{\frac{a^3}{2\sqrt{2}}-\frac{3}{2}a^2\left(\frac{x}{r}\right)+\frac{3}{\sqrt{2}}a\left(\frac{x}{r}\right)^2-\left(\frac{x}{r}\right)^3\right\}$$

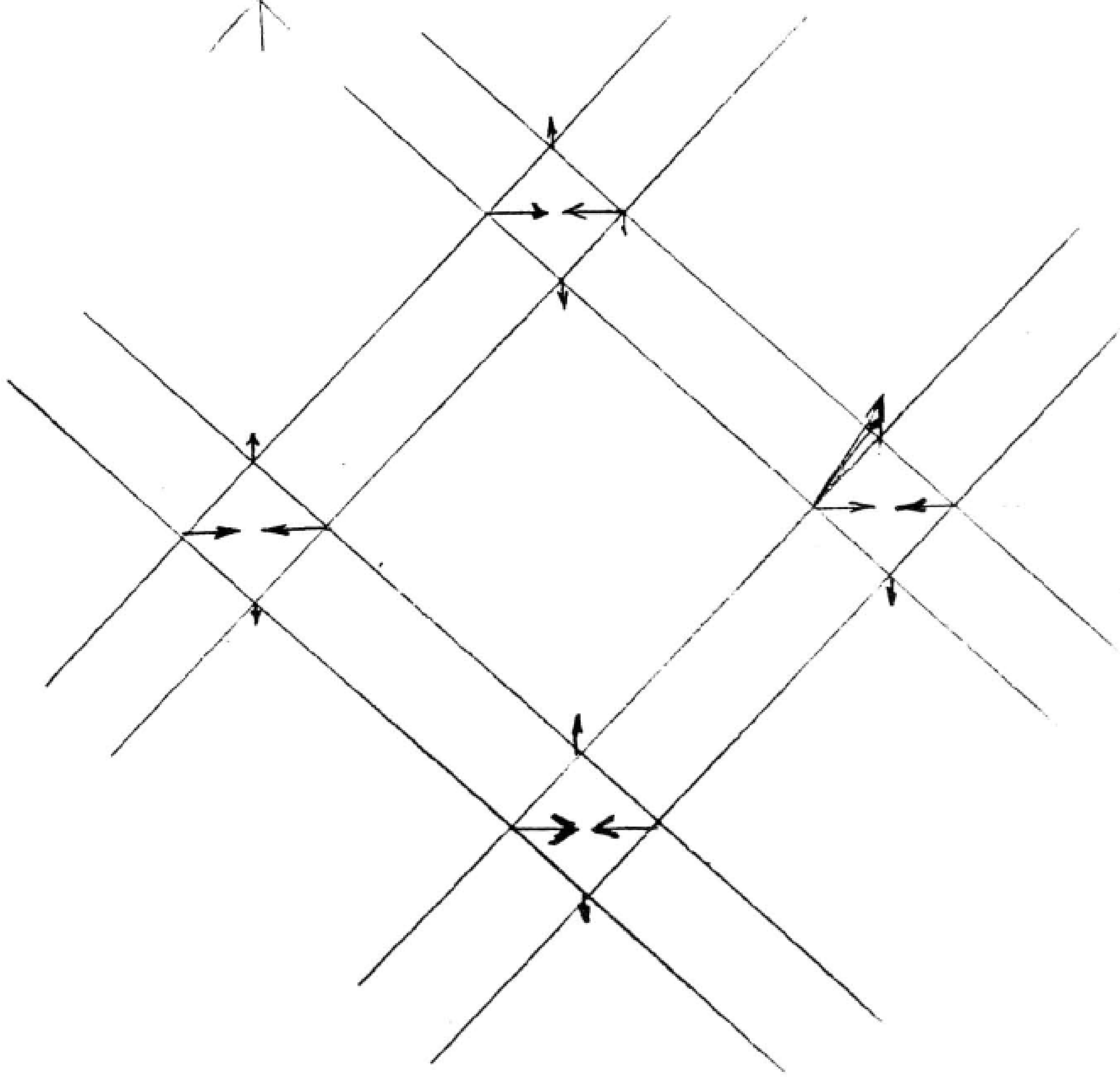
$$= \frac{a^3}{3\sqrt{2}}-\frac{3}{2\sqrt{2}}a\left(\frac{x}{r}\right)^2+\frac{5}{6}\left(\frac{x}{r}\right)^3$$

The u, v in region (IV) can be expressed as 120,



$$u = \frac{p}{6}\left(\frac{a}{b}\right)^3\left(\frac{x}{r}\right)^3 - p\underline{\underline{k}}\left(\frac{x}{p}\right)$$

$$v = \left(\frac{a}{b}\right)^3\left\{\frac{p}{2}\left(b^2 - \frac{x^2}{r^2}\right)\theta - \frac{p}{3}\theta^3\right\}$$



The u, v in the rectangular region can be obtained 121)
by returning the ◇ figure 90° counter-clockwise +
than make v negative.

$$u = -\left\{ \frac{\gamma}{6}\,\theta^3 - \frac{\gamma k}{2}\,\theta \right\}$$

$$v = -\left\{ \frac{\gamma}{2}\left(a^2 - \theta^2\right)\left(\frac{x}{r}\right) - \frac{\gamma}{3}\left(\frac{x}{r}\right)^3 \right\}$$

We then shrink one side by the ratio $\left(\frac{b}{a}\right)$

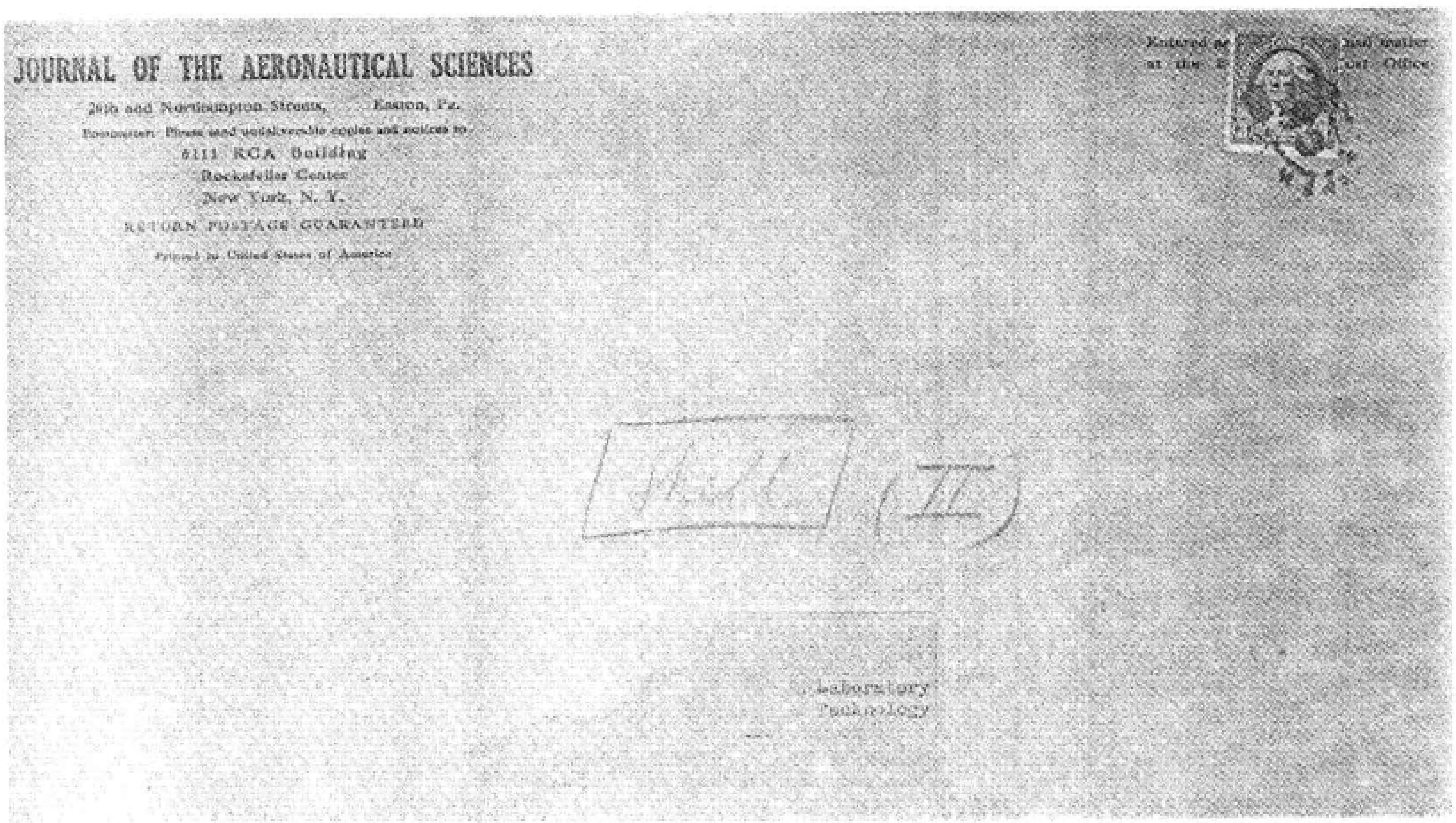
$$\theta = \left(\frac{a}{b}\right)\frac{s}{\sqrt{2}} - \frac{t}{\sqrt{2}}$$

$$\frac{x}{r} = \left(\frac{a}{b}\right)\frac{s}{\sqrt{2}} + \frac{t}{\sqrt{2}}$$

$$\bar{u} = \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}} \qquad \bar{v} = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$$

Section 2

Shell (Ⅱ) Collapse of Slightly Curved Circular Plate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Collapse of a Slightly Curved Circular Plate

Ref: C.B. Biezeno: Über die Bestimmung der „Durchschlagskraft" einer schwachgekrümmten, kreisförmigen Platte. ZAMM, Bd. 15, S. 10-22 (1935)

In this calculation, it is assumed that the plate are fixed at the boundary and the acting force is a pressure "p" uniformly distributed over the whole area. Then the equilibrium condition for obtaining D is

$$2\pi(x+u)R\sin(\varphi+\psi) - 2\pi(x+u)D\cos(\phi+\psi) = -\pi(x+u)^2 p$$

$$\therefore \quad 2R(\phi+\psi) - 2D = -(x+u)p$$

$$\boxed{D = R(\phi+\psi) + \frac{(x+u)}{2}p}$$

Substituting into the two equilibrium condition we have

$$\begin{cases} \left[R(x+u)\cos(\phi+\psi)\right]' - T(1+u') + \left\{\left[R(\phi+\psi) + \frac{(x+u)}{2}p\right](\phi+\psi)\cdot(x+u)\right\}' = 0 \\ -\left[M_t(x+u)\right]' + M_r(1+u') + \left[R(\phi+\psi) + \frac{(x+u)}{2}p\right](x+u)(1+u') = 0. \end{cases}$$

$$\begin{cases} R + xR' - T + \frac{p}{2}\left[2x(\phi+\psi) + x^2(\frac{1}{5} + \psi')\right] = 0. \\ -(M_t x)' + M_r + R(\phi+\psi)x + \frac{p}{2}x^2 = 0 \end{cases}$$

$$\psi = C\left(\frac{x}{r}\right)\left(1 - \frac{x^2}{r^2}\right)$$

[See Love's Elasticity, p. 490]

$$\therefore\quad \chi_1 = u' + \frac{1}{m}\frac{u}{x} + C\phi\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2 \qquad 193)$$

$$= u' + \frac{1}{m}\frac{u}{x} + C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2$$

$$\chi_2 = \frac{C}{r}\left\{1-\frac{x^2}{r^2} + \frac{x}{r}\left(-\frac{2x}{r}\right)\right\} + \frac{1}{m}\frac{C}{r}\left(1-\frac{x^2}{r^2}\right)$$

$$= \frac{C}{r}\left\{1-3\frac{x^2}{r^2} + \frac{1}{m}\left(1-\frac{x^2}{r^2}\right)\right\}$$

$$= \frac{C}{r}\left\{\left(1+\frac{1}{m}\right) - \frac{x^2}{r^2}\left(3+\frac{1}{m}\right)\right\}$$

$$\chi_3 = \frac{u}{x} + \frac{1}{m}\left[u' + C\frac{x}{\rho}\cdot\frac{x}{r}\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2\right]$$

$$\chi_4 = \frac{C}{r}\left(1-\frac{x^2}{r^2}\right) + \frac{C}{mr}\left\{1-3\frac{x^2}{r^2}\right\}$$

$$= \frac{C}{r}\left\{\left(1+\frac{1}{m}\right) - \left(1+\frac{3}{m}\right)\frac{x^2}{r^2}\right\}$$

$$R = \frac{m^2Eh}{m^2-1}\left\{u' + \frac{1}{m}\frac{u}{x} + C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2\right\}$$

$$R' = \frac{m^2Eh}{m^2-1}\left\{u'' + \frac{1}{m}\frac{u'}{x} - \frac{1}{m}\frac{u}{x^2} + \cancel{C\left(\frac{1}{\rho}\right)\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right.} \right.$$

$$C\frac{1}{\rho r}\left\{2x - 4\frac{x^3}{r^2}\right\} + \frac{C^2}{2r^2}\,2x\left(1-\frac{x^2}{r^2}\right)^2$$

$$\left. + \frac{C^2}{2}\frac{x^2}{r^2}\,2\left(1-\frac{x^2}{r^2}\right)\left(-\frac{2x}{r^2}\right)\right\}$$

$$= \frac{m^2Eh}{m^2-1}\left\{u'' + \frac{1}{m}\frac{u'}{x} - \frac{1}{m}\frac{u}{x^2} + \frac{2C}{\rho}\left(\frac{x}{r}\right)\left(1-2\frac{x^2}{r^2}\right) + \frac{C^2}{r}\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right)^2\right.$$

$$\left. - \frac{2C^2}{r}\left(\frac{x}{r}\right)^3\left(1-\frac{x^2}{r^2}\right)\right\}$$

$$R' = \frac{m^2 E h}{m^2-1}\left\{ u'' + \frac{1}{m}\frac{u'}{x} - \frac{1}{m}\frac{u}{x^2} + \frac{2C}{\rho}\left(\frac{x}{r}\right)\left(1-2\frac{x^2}{r^2}\right) + \frac{C^2}{r}\left(\frac{x}{r}\right)\left(1-4\frac{x^2}{r^2}+3\frac{x^4}{r^4}\right)\right\}$$

$$T = \frac{m^2 E h}{m^2-1}\left\{\frac{u}{x} + \frac{1}{m}\left[u' + C\frac{x}{\rho}\frac{x}{r}\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2\right]\right\}$$

$$\therefore \quad u' + \cancel{\frac{1}{m}\frac{u}{x}} + C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2$$

$$+ \; x u'' + \cancel{\frac{1}{m}} u' - \cancel{\frac{1}{m}\frac{u}{x}} + 2C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-2\frac{x^2}{r^2}\right) + C^2\frac{x^2}{r^2}\left(1-4\frac{x^2}{r^2}+3\frac{x^4}{r^4}\right)$$

$$-\left\{\frac{u}{x} + \frac{1}{m}\left[\cancel{u'} + C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2\right]\right\}$$

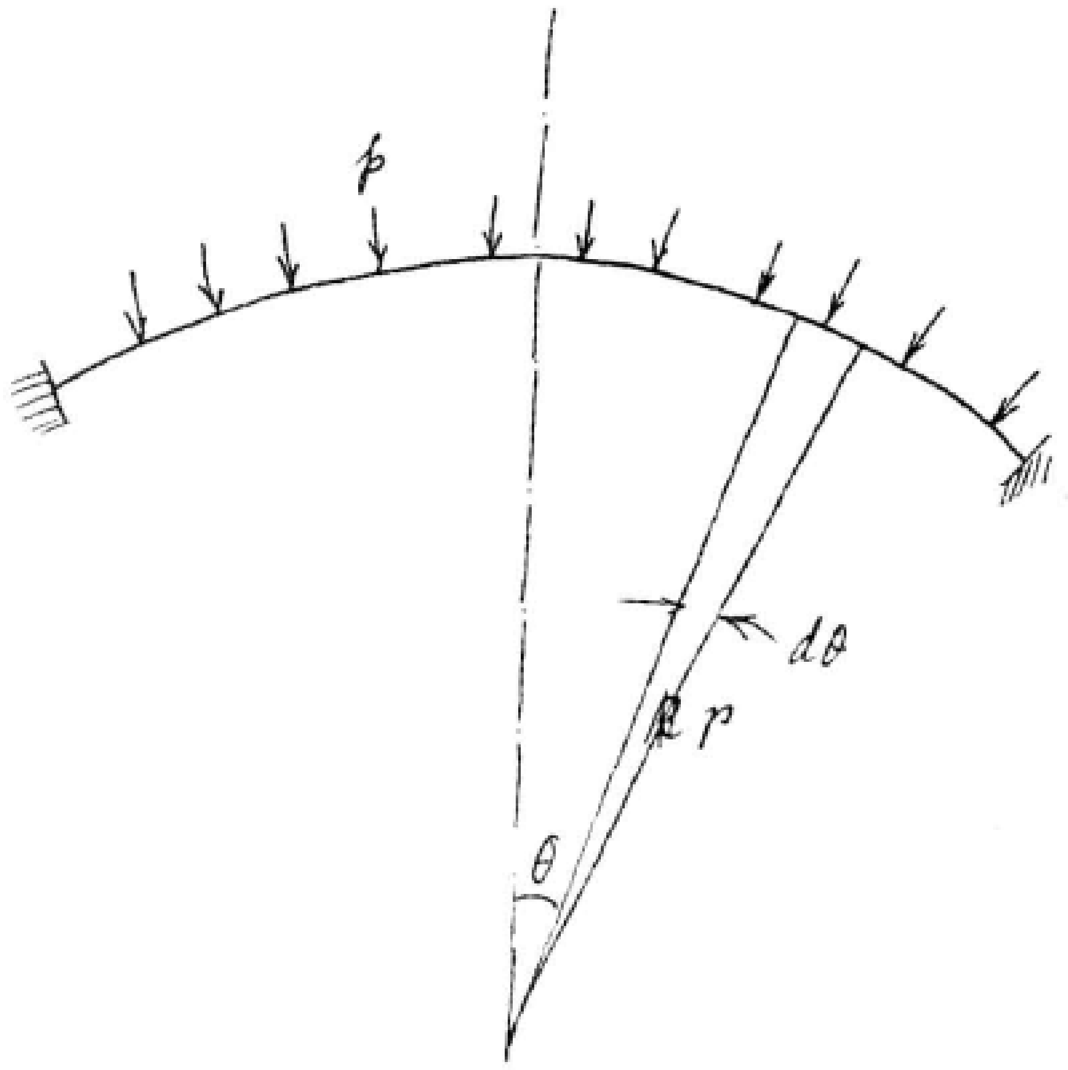
$$+ (m^2-1)\frac{p}{2}\left[2x\left(\frac{x}{\rho} + C\frac{x}{r}\left(1-\frac{x^2}{r^2}\right)\right) + x^2\left(\frac{1}{\rho} + \frac{C}{r}\left(1-3\frac{x^2}{r^2}\right)\right)\right] = 0.$$

$$\cancel{x u'' + u'\left(1+\frac{1}{m}\right)} - \frac{u}{x} =$$

$$x u'' + u' - \frac{u}{x} = \left\{C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-\frac{x^2}{r^2}\right) + \frac{C^2}{2}\frac{x^2}{r^2}\left(1-\frac{x^2}{r^2}\right)^2\right\}\left(\frac{1}{m}-1\right)$$

$$- 2C\left(\frac{x}{\rho}\right)\left(\frac{x}{r}\right)\left(1-2\frac{x^2}{r^2}\right) + C^2\frac{x^2}{r^2}\left(1-4\frac{x^2}{r^2}+3\frac{x^4}{r^4}\right)$$

$$- \frac{p}{2}x\left(\frac{m^2-1}{m^2 E h}\right)\left[2\left(\frac{x}{\rho}\right) + C\frac{x}{r}\left[3-5\frac{x^2}{r^2}\right]\right] \neq 0.$$



Let $\cancel{\bar{\theta} = \theta + \frac{u}{R}}$ $\qquad \bar{\theta} = \theta + \frac{u}{r}$

$\cancel{\bar{R} = R + Rw = R(1-w)}$ $\qquad \bar{r} = r - w$

The original length of the element $ds = r\,d\theta$

The new length of the elemegment

$$= \sqrt{\bar{r}^2 (d\bar{\theta})^2 + (d\bar{r})^2} = \sqrt{(r-w)^2\left(1+\frac{1}{r}\frac{du}{d\theta}\right)^2 + \left(-\frac{dw}{d\theta}\right)^2}\; d\theta$$

$$= r\,d\theta \sqrt{\left(1-\frac{w}{r}\right)^2\left(1+\frac{1}{r}\frac{du}{d\theta}\right)^2 + \frac{1}{r^2}\left(\frac{dw}{d\theta}\right)^2}$$

$$\varepsilon_\theta = \sqrt{\left(1-\frac{w}{r}\right)^2\left(1+\frac{1}{r}\frac{du}{d\theta}\right)^2 + \frac{1}{r^2}\left(\frac{dw}{d\theta}\right)^2} - 1$$

$$= \left\{\left[1 - 2\frac{w}{r} + \left(\frac{w}{r}\right)^2\right]\left[1 + \frac{2}{r}\frac{du}{d\theta} + \frac{1}{r^2}\left(\frac{du}{d\theta}\right)^2\right] + \frac{1}{r^2}\left(\frac{dw}{d\theta}\right)^2\right\}^{\frac{1}{2}} - 1$$

$$= \left\{1 - 2\frac{w}{r} + \left(\frac{w}{r}\right)^2 + \frac{2}{r}\frac{du}{d\theta} - \frac{4}{r^2} w \frac{du}{d\theta} + \frac{2}{r}\frac{du}{d\theta}\cdot\left(\frac{w}{r}\right)^2 + \frac{1}{r^2}\left(\frac{du}{d\theta}\right)^2 - \cdots + \frac{1}{r^2}\left(\frac{dw}{d\theta}\right)^2\right\}^{\frac{1}{2}} - 1$$

Retaining only the important terms, we have 196)

$$\varepsilon_1 = \frac{1}{r}\frac{du}{d\theta} - \frac{w}{r} + \frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2$$

8 The distance between the points and the axis of symmetry is $r \sin\theta$,

$$\varepsilon_2 = \frac{\bar{r}\sin\bar{\theta} - r\sin\theta}{r\sin\theta}$$

$$= \left(1-\frac{w}{r}\right)\frac{\sin\bar{\theta}}{\sin\theta} - 1 = \left(1-\frac{w}{r}\right)\frac{\sin\left(\theta+\frac{u}{r}\right)}{\sin\theta} - 1$$

$$= \left(1-\frac{w}{r}\right)\left[1 + \frac{u}{r}\cot\theta\right] - 1$$

$$= -\frac{w}{r} + \frac{u}{r}\cot\theta - \left(\frac{w}{r}\right)\frac{u}{r}\cot\theta$$

Hence the extensional energy per unit area

$$2W_e = \frac{Et}{(1-\mu^2)}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\mu)\left(\varepsilon_1\varepsilon_2 - \tfrac{1}{4}\theta^2\right)\right]$$

$$= \frac{Et}{(1-\mu^2)}\left[\left\{\frac{1}{r}\frac{du}{d\theta} - \frac{w}{r} + \frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2 - \frac{w}{r} + \frac{u}{r}\cot\theta - \left(\frac{w}{r}\right)\frac{u}{r}\cot\theta\right\}^2\right.$$

$$\left. -2(1-\mu)\left[\frac{1}{r}\frac{du}{d\theta} - \frac{w}{r} + \frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right]\left[-\frac{w}{r} + \frac{u}{r}\cot\theta - \left(\frac{w}{r}\right)\frac{u}{r}\cot\theta\right]\right]$$

$$= \frac{Et}{(1-\mu^2)}\left[\left\{\frac{1}{r}\frac{du}{d\theta} - \frac{2w}{r} + \frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2 + \frac{u}{r}\cot\theta\left(1-\frac{w}{r}\right)\right\}^2\right.$$

$$\left. - 2(1-\mu)\left\{\frac{1}{r}\frac{du}{d\theta} - \frac{w}{r} + \frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right\}\left\{\frac{u}{r}\cot\theta\left(1-\frac{w}{r}\right) - \frac{w}{r}\right\}\right]$$

Total extensional energy

$$2W_e = \frac{Et}{(1-\mu^2)} 2\pi r^2 \int_0^{\Theta} \sin\theta \Big[\Big\{ \frac{1}{r}\frac{du}{d\theta} - \frac{2w}{r} + \frac{1}{2r^2}\Big(\frac{dw}{d\theta}\Big)^2 + \frac{u}{r}\cos\theta \Big\}^2$$

$$- 2(1-\mu)\Big\{ \frac{1}{r}\frac{du}{d\theta} - \frac{w}{r} + \frac{1}{2r^2}\Big(\frac{dw}{d\theta}\Big)^2 \Big\}\Big\{ \frac{u}{r}\cos\theta - \frac{w}{r} \Big\} \Big] d\theta$$

The total bending energy

$$\cancel{2W_b = \frac{E\,t^3}{12(1-\mu^2)} \Big[\Big\{ \frac{d^2w}{a^2 d\theta^2} + \frac{du}{a^2 d\theta} \Big\}}$$

$$2W_b = \frac{E\,t^3}{12(1-\mu^2)} 2\pi r^2 \int_0^{\Theta} \sin\theta \Big[\Big\{ \frac{d^2w}{r^2 d\theta^2} + \frac{du}{r^2 d\theta} + \Big(\frac{u}{r^2} + \frac{dw}{r^2 d\theta} \Big)\cos\theta \Big\}^2$$

$$- 2(1-\mu)\Big\{ \frac{d^2w}{r^2 d\theta^2} + \frac{du}{r^2 d\theta} \Big\}\Big\{ \frac{u}{r^2} + \frac{dw}{r^2 d\theta} \Big\}\cos\theta \Big] d\theta$$

The potential energy is expressed as the "p" times the volume

$$p\,\frac{1}{3}\int_0^{\Theta} 2\pi \bar{r}\sin\bar{\theta}\;\bar{r}\cdot\bar{r}\; d\bar{\theta}$$

$$= \frac{2\pi p r^3}{3} \int_0^{\Theta} \Big(1-\frac{w}{r}\Big)^3 \sin\bar{\theta}\, d\bar{\theta} = \frac{2\pi p r^3}{3} \int_0^{\Theta} \overset{(1-\frac{w}{r})^3}{} \sin\Big(\theta + \frac{u}{r}\Big)\Big(1 + \frac{1}{r}\frac{du}{d\theta}\Big) d\theta$$

$$= \frac{2\pi p r^3}{3} \int_0^{\Theta} \Big(1-\frac{w}{r}\Big)^3 \Big(1 + \frac{1}{r}\frac{du}{d\theta}\Big)\Big(\sin\theta + \frac{u}{r}\cos\theta\Big) d\theta$$

$$\cong \frac{2\pi p r^3}{3} \int_0^{\Theta} \Big(1 - \frac{3w}{r}\Big)\Big(1 + \frac{1}{r}\frac{du}{d\theta}\Big)\Big(\sin\theta + \frac{u}{r}\cos\theta\Big) d\theta$$

$$\cong \frac{2\pi p r^3}{3}\int_0^{\Theta}\left\{\sin\theta\left(1-\frac{3w}{r}+\frac{1}{r}\frac{du}{d\theta}\right)+\frac{u}{r}\cos\theta\right\}d\theta$$

$$\frac{V}{2\pi r^3}=\frac{E}{(1-\mu^2)}\Bigg\{\left(\frac{t}{r}\right)\int_0^{\Theta}\sin\theta\Bigg[\overset{\mathrm{I}}{\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{2w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2+\frac{u}{r}\cot\theta\right\}^2}-$$

$$-2(1-\mu)\overset{\mathrm{II}}{\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right\}\left\{\frac{u}{r}\cot\theta-\frac{w}{r}\right\}}\Bigg]d\theta$$

$$+\frac{1}{12}\left(\frac{t}{r}\right)^3\int_0^{\Theta}\sin\theta\Bigg[\overset{\mathrm{III}}{\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}+\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\cot\theta\right\}^2}-2(1-\mu)\overset{\mathrm{IV}}{\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}\right\}\left\{\frac{u}{r}+\frac{dw}{rd\theta}\right\}\cot\theta}\Bigg]d\theta\Bigg\}$$

$$-\frac{p}{3}\int_0^{\Theta}\sin\theta\left\{\cancel{\sin\theta}\left(-\overset{\mathrm{I}}{\frac{3w}{r}}+\frac{1}{r}\frac{du}{d\theta}\right)+\frac{u}{r}\cot\theta\right\}d\theta$$

Term not correct!

how minimizing for $\left(\frac{u}{r}\right)$

$$\left(\frac{t}{r}\right)\Bigg[2\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{2w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2+\frac{u}{r}\cot\theta\right\}\cot\theta$$

$$-2(1-\mu)\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right\}\cot\theta-2\frac{d}{d\theta}\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{2w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2+\frac{u}{r}\cot\theta\right\}$$

$$+2(1-\mu)\frac{d}{d\theta}\left\{\frac{u}{r}\cot\theta-\frac{w}{r}\right\}$$

$$+\frac{1}{12}\left(\frac{t}{r}\right)^3\Bigg[2\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}+\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\cot\theta\right\}\cot\theta-2(1-\mu)\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}\right\}\cot\theta$$

$$-2\frac{d}{d\theta}\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}+\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\cot\theta\right\}+2(1-\mu)\frac{d}{d\theta}\left[\cot\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\right]\Bigg]$$

$$-\frac{p}{3}\Big[\cot\theta\ \cancel{\theta}\ -$$

Now minimizing for $\left(\frac{u}{r}\right)$

$$\frac{\left(\frac{t}{r}\right)}{(1-\mu^2)}\Big[\cos\theta\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{2w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2+\frac{u}{r}\cot\theta\right\}$$

$$-(1-\mu)\cos\theta\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right\}-\frac{d}{d\theta}\left\{\frac{\sin\theta}{r}\frac{du}{d\theta}-2\sin\theta\frac{w}{r}+\frac{\sin\theta}{2r^2}\left(\frac{dw}{d\theta}\right)^2+\frac{u}{r}\cos\theta\right\}$$

$$+(1-\mu)\frac{d}{d\theta}\left\{\frac{u}{r}\cos\theta-\sin\theta\frac{w}{r}\right\}$$

$$+\frac{\left(\frac{t}{r}\right)^3}{12(1-\mu^2)}\Big[\cos\theta\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}+\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\cot\theta\right\}-(1-\mu)\cos\theta\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}\right\}$$

$$-\frac{d}{d\theta}\left\{\sin\theta\frac{d^2w}{rd\theta^2}+\sin\theta\frac{du}{rd\theta}+\cos\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\right\}+(1-\mu)\frac{d}{d\theta}\left\{\cos\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\right\}\Big]\Big]$$

$$-\frac{p}{3}\Big[\cos\theta-\cos\theta\Big]=0$$

Therefore:

$$\cos\theta\left\{\cancel{\frac{1}{r}\frac{du}{d\theta}}-\frac{2w}{r}+\cancel{\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2}+\frac{u}{r}\cot\theta\right\}-(1-\mu)\cos\theta\left\{\frac{1}{r}\frac{du}{d\theta}-\frac{w}{r}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right\}$$

$$-\cancel{\frac{\cos\theta}{r}\frac{du}{d\theta}}+2\cos\theta\frac{w}{r}-\cancel{\frac{\cos\theta}{2r^2}\left(\frac{dw}{d\theta}\right)^2}+\frac{u}{r}\sin\theta-\left\{\frac{\sin\theta}{r}\frac{d^2u}{d\theta^2}-2\frac{\sin\theta}{r}\frac{dw}{d\theta}\right.$$
$$\left.+\frac{\sin\theta}{r^2}\frac{dw}{d\theta}\frac{d^2w}{d\theta^2}+\frac{\cos\theta}{r}\frac{du}{d\theta}\right\}$$

$$+(1-\mu)\left\{-\sin\theta\frac{u}{r}+\cancel{\frac{\cos\theta}{r}\frac{du}{d\theta}}-\cancel{\cos\theta\frac{w}{r}}-\frac{\sin\theta}{r}\frac{dw}{d\theta}\right\}\Big]$$

$$+\frac{1}{12}\left(\frac{t}{r}\right)^2\Big[\cos\theta\left\{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}+\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\cot\theta\right\}-(1-\mu)\cos\theta\left\{\cancel{\frac{d^2w}{rd\theta^2}+\frac{du}{rd\theta}}\right\}$$

$$-\left\{\cancel{\cos\theta\frac{d^2w}{rd\theta^2}}+\cancel{\cos\theta\frac{du}{rd\theta}}-\sin\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\right\}-\left\{\sin\theta\frac{d^3w}{rd\theta^3}+\sin\theta\frac{d^2u}{rd\theta^2}+\cos\theta\left(\frac{1}{r}\frac{du}{d\theta}+\frac{d^2w}{rd\theta^2}\right)\right\}$$

$$+(1-\mu)\left\{-\sin\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)+\cos\theta\left(\cancel{\frac{1}{r}\frac{du}{d\theta}+\frac{d^2w}{rd\theta^2}}\right)\right\}=0 \qquad 130)$$

$$\frac{u}{r}\left[\frac{\cos^2\theta+\sin^2\theta}{\sin\theta}\right]-(1-\mu)\cos\theta\left\{\cancel{\frac{1}{r}}\frac{du}{d\theta}+\frac{1}{2r^2}\left(\frac{dw}{d\theta}\right)^2\right\}$$

$$-\left\{\sin\theta\frac{d^2u}{rd\theta^2}-2\frac{\sin\theta}{r}\frac{dw}{d\theta}+\frac{\sin\theta}{r^2}\frac{dw}{d\theta}\frac{d^2w}{d\theta^2}+\frac{\cos\theta}{r}\frac{du}{d\theta}\right\}$$

$$+(1-\mu)\left\{-\sin\theta\frac{u}{r}+\cancel{\frac{\cos\theta}{r}\frac{du}{d\theta}}-\frac{\sin\theta}{r}\frac{dw}{d\theta}\right\}$$

$$+\frac{1}{12}\left(\frac{t}{r}\right)^2\left[\frac{1}{\sin\theta}\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)+\cancel{\sin\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)}-\sin\theta\frac{d^3w}{rd\theta^3}-\sin\theta\frac{d^2u}{rd\theta^2}\right.$$
$$\left.-\cos\theta\left(\frac{1}{r}\frac{du}{d\theta}+\frac{d^2w}{rd\theta^2}\right)-(1-\mu)\sin\theta\left(\frac{u}{r}+\frac{dw}{rd\theta}\right)\right]=0.$$

Write $\frac{u}{r}\simeq u, \qquad \frac{w}{r}\sim w,$

$$\underline{u}-(1-\mu)\left\{\underline{\frac{\sin\theta\cos\theta}{2}\left(\frac{dw}{d\theta}\right)^2}+\underline{\sin^2\theta\cdot u}+\sin^2\theta\frac{dw}{d\theta}\right\}$$

$$-\underline{\sin^2\theta\frac{d^2u}{d\theta^2}}+2\sin^2\theta\frac{dw}{d\theta}-\sin^2\theta\frac{dw}{d\theta}\frac{d^2w}{d\theta^2}-\underline{\sin\theta\cos\theta\frac{du}{d\theta}}$$

$$+\frac{1}{12}\left(\frac{t}{r}\right)^2\left[\underline{u}+\frac{dw}{d\theta}-\sin^2\theta\frac{d^3w}{d\theta^3}-\underline{\sin^2\theta\frac{d^2u}{d\theta^2}}-\sin\theta\cos\theta\left(\underline{\frac{du}{d\theta}}+\frac{d^2w}{d\theta^2}\right)\right.$$
$$\left.-(1-\mu)\sin^2\theta\left(\underline{u}+\frac{dw}{d\theta}\right)\right]=0.$$

If we put $\frac{1}{12}\left(\frac{t}{r}\right)^2=\alpha$

$$(1+\alpha)\left\{\sin^2\theta\frac{d^2u}{d\theta^2}+\sin\theta\cos\theta\frac{du}{d\theta}+[-1+\sin^2\theta(1-\mu)]u\right\}$$

$$=\sin^2\theta\frac{dw}{d\theta}\left[2-\frac{d^2w}{d\theta^2}\right]-(1-\mu)(1+\alpha)\sin^2\theta\frac{dw}{d\theta}-(1-\mu)\frac{\sin\theta\cos\theta}{2}\left(\frac{dw}{d\theta}\right)^2$$

$$+\alpha\left\{\frac{dw}{d\theta}-\sin^2\theta\frac{d^3w}{d\theta^3}-\sin\theta\cos\theta\frac{d^2w}{d\theta^2}\right\}$$

$\cancel{\frac{\partial t}{\sin^2\theta}}$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{du}{d\theta}\right)+\left\{(1-\mu)-\frac{1}{\sin^2\theta}\right\}u$$

$$=\frac{1}{\sin^2\theta\cdot(1+\alpha)}\left\{\sin^2\theta\left(2-\frac{d^2w}{d\theta^2}\right)\frac{dw}{d\theta}-(1+\alpha)(1-\mu)\sin^2\theta\frac{dw}{d\theta}\right.$$
$$\left.-(1-\mu)\frac{\sin\theta\cos\theta}{2}\left(\frac{dw}{d\theta}\right)^2+\alpha\left[\frac{dw}{d\theta}-\sin^2\theta\frac{d^3w}{d\theta^3}-\sin\theta\cos\theta\frac{d^2w}{d\theta^2}\right]\right\}$$

If we put

$$w=C[\cos\theta-k\cos 2\theta]$$
$$\frac{dw}{d\theta}=C[-\sin\theta+2k\sin 2\theta]$$
$$\frac{d^2w}{d\theta^2}=C[-\cos\theta+4k\cos 2\theta]$$
$$\frac{d^3w}{d\theta^3}=C[\sin\theta-8k\sin 2\theta]$$

$$\left(2-\frac{d^2w}{d\theta^2}\right)\frac{dw}{d\theta}=C\left[2+C(\cos\theta-4k\cos 2\theta)\right]\left[-\sin\theta+2k\sin 2\theta\right]$$

$$\frac{\cos\theta}{2\sin\theta}\left(\frac{dw}{d\theta}\right)^2=\frac{\sin\theta\cos\theta}{2}C^2\left[1-4k\cos\theta\right]^2$$

$$\frac{1}{\sin^2\theta}\frac{dw}{d\theta}=-\frac{C}{\sin\theta}\left[1-4k\cos\theta\right]$$

$$\frac{\cos\theta}{\sin\theta}\frac{d^2w}{d\theta^2}=C\frac{\cos\theta}{\sin\theta}\left[-\cos\theta+4k\cos 2\theta\right]$$

It is better to obtain the equation for u separately from the equilibrium consideration.

$$\left.\begin{aligned} \varepsilon_1 &= \frac{du}{d\theta} - w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 \\ \varepsilon_2 &= u \cdot \cot\theta - w \end{aligned}\right\}$$

$$N_x = -\frac{pr}{2} + \frac{Et}{1-\mu^2}\left\{\frac{du}{d\theta} - w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 + \mu\left[u \cdot \cot\theta - w\right]\right\}$$

$$N_y = -\frac{pr}{2} + \frac{Et}{1-\mu^2}\left\{u \cdot \cot\theta - w + \mu\left[\frac{du}{d\theta} - w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right]\right\}$$

$$M_x = -\frac{D}{r}\left\{\frac{d^2w}{d\theta^2} + \frac{du}{d\theta} + \mu\left[u + \frac{dw}{d\theta}\right]\cot\theta\right\}$$

$$M_y = -\frac{D}{r}\left\{\left(u + \frac{dw}{d\theta}\right)\cot\theta + \mu\left[\frac{d^2w}{d\theta^2} + \frac{du}{d\theta}\right]\right\}$$

The equations of equilibrium is

$$\cancel{\frac{dN_x}{d\theta} + (N_x - N_y)\cot\theta - Q_x + N_y\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) - Q_x\left(\frac{d^2w}{d\theta^2}\right.}$$

$$\frac{dN_x}{d\theta} + (N_x - N_y)\cot\theta - Q_x\left(1 + w + \frac{d^2w}{d\theta^2}\right) + N_y\left(-u + \frac{dw}{d\theta}\right) = 0$$

$$\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x + N_y + rp + N_x\left(\frac{d^2w}{d\theta^2} + \frac{du}{d\theta}\right) + N_y\left(u + \frac{dw}{d\theta}\right)\cot\theta = 0$$

$$\frac{1}{r}\left\{\frac{dM_x}{d\theta} + (M_x - M_y)\cot\theta + M_y\left(\frac{u}{\cancel{a}} + \frac{dw}{d\theta}\right)\right\} = Q_x$$

133)

$$\boxed{\frac{dN_x}{d\theta}+(N_x-N_y)\cot\theta-\frac{\left(1+w+\frac{d^2w}{d\theta^2}\right)}{r}\left\{\frac{dM_x}{d\theta}+(M_x-M_y)\cot\theta+M_y\left(u+\frac{dw}{d\theta}\right)\right\} + N_y\left(u+\frac{dw}{d\theta}\right)=0}$$

$$\cancel{\frac{1}{r}\left\{\frac{d^2M_x}{d\theta^2}+\left(\frac{dM_x}{d\theta}-\frac{dM_y}{d\theta}\right)\cot\theta - (M_x-M_y)\csc^2\theta+\frac{dM_y}{d\theta}\left(u+\frac{dw}{d\theta}\right)\right.}$$

$$\cancel{\left.+M_y\left(\frac{du}{d\theta}+\frac{d^2w}{d\theta^2}\right)\right\}+\frac{\cot\theta}{r}\left\{\frac{dM_x}{d\theta}+(M_x-M_y)\cot\theta+M_y\left(\frac{u}{a}+\frac{dw}{d\theta}\right)\right\}}$$

$$\cancel{+N_x+N_y+rp+N_x\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)+N_y\left(u+\frac{dw}{d\theta}\right)\cot\theta=0.}$$

$$\frac{1}{r}\left\{\frac{d^2M_x}{d\theta^2}+\left(2\frac{dM_x}{d\theta}-\frac{dM_y}{d\theta}\right)\cot\theta-(M_x-M_y)+\frac{dM_y}{d\theta}\left(u+\frac{dw}{d\theta}\right)+M_y\left(u+\frac{dw}{d\theta}+\frac{du}{d\theta}\cot\theta+\frac{d^2w}{d\theta^2}\cot\theta\right)\right\}$$

$$+N_x+N_y+rp+N_x\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)+N_y\left(u+\frac{dw}{d\theta}\right)\cot\theta=0.$$

Only the first equation is to be used as a means to find u.

$$\frac{E\left(\frac{t}{r}\right)}{(1-\mu^2)}\left[\left\{\frac{d^2u}{d\theta^2}-\frac{dw}{d\theta}+\frac{dw}{d\theta}\frac{d^2w}{d\theta^2}+\mu\left[\frac{du}{d\theta}\cot\theta-u\csc^2\theta-\frac{dw}{d\theta}\right]\right\}\right.$$

$$\left.+(1-\mu)\cot\theta\left\{\frac{du}{d\theta}-w+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2-u\cot\theta+w\right\}\right]$$

$$+\frac{E\left(\frac{t}{r}\right)^3}{12(1-\mu^2)}\left(1+w+\frac{d^2w}{d\theta^2}\right)\left\{\frac{d^3w}{d\theta^3}+\frac{d^2u}{d\theta^2}+\mu\left[\frac{du}{d\theta}+\frac{d^2w}{d\theta^2}\right]\cot\theta-\mu\left[u+\frac{dw}{d\theta}\right]\csc^2\theta\right.$$

$$\left.+(1-\mu)\cot\theta\left[\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}-\left(u+\frac{dw}{d\theta}\right)\cot\theta\right]+\left(u+\frac{dw}{d\theta}\right)\left[\left(u+\frac{du}{d\theta}\right)\cot\theta+\mu\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)\right]\right\}$$

$$+\left(u+\frac{dw}{d\theta}\right)\left[-\frac{p}{2}+\frac{E\left(\frac{t}{r}\right)}{(1-\mu^2)}\left\{u\cdot\cot\theta-w+\mu\left[\frac{du}{d\theta}-w+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right]\right\}\right]=0 \qquad (134)$$

Let $\quad \dfrac{\left(\frac{t}{r}\right)^2}{12}=\alpha, \qquad \dfrac{p(1-\mu^2)}{2E\left(\frac{t}{r}\right)}=\phi$

$$\frac{d^2u}{d\theta^2}-\frac{dw}{d\theta}+\frac{dw}{d\theta}\frac{d^2w}{d\theta^2}+\mu\left(\frac{du}{d\theta}\cdot\cot\theta-u\csc^2\theta-\frac{dw}{d\theta}\right)+(1-\mu)\cot\theta\left\{\frac{du}{d\theta}-u\cot\theta+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right\}$$

$$+\alpha\left\{\frac{d^3w}{d\theta^3}+\frac{d^2u}{d\theta^2}+\mu\left[\left(\frac{du}{d\theta}+\frac{d^2w}{d\theta^2}\right)\cot\theta-\left(u+\frac{dw}{d\theta}\right)\csc^2\theta\right]\right.$$

$$\left.+(1-\mu)\cot\theta\left[\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}-\left(u+\frac{dw}{d\theta}\right)\cot\theta\right]+\left(u+\frac{dw}{d\theta}\right)\left[\left(u+\frac{dw}{d\theta}\right)\cot\theta+\mu\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)\right]\right\}$$

$$-\phi\left(u+\frac{dw}{d\theta}\right)+\left(u+\frac{dw}{d\theta}\right)\left\{u\cot\theta-w+\mu\left[\frac{du}{d\theta}-w+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right]\right\}=0.$$

Linearize in u,

$$\frac{d^2u}{d\theta^2}-\frac{dw}{d\theta}+\frac{dw}{d\theta}\frac{d^2w}{d\theta^2}+\mu\left(\frac{du}{d\theta}\cot\theta-u\csc^2\theta-\frac{dw}{d\theta}\right)+(1-\mu)\cot\theta\left\{\frac{du}{d\theta}-u\cot\theta+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right\}$$

$$+\alpha\left\{\frac{d^3w}{d\theta^3}+\frac{d^2u}{d\theta^2}+\mu\left[\left(\frac{du}{d\theta}+\frac{d^2w}{d\theta^2}\right)\cot\theta-\left(u+\frac{dw}{d\theta}\right)\csc^2\theta\right]\right.$$

$$+(1-\mu)\cot\theta\left[\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}-\left(u+\frac{dw}{d\theta}\right)\cot\theta\right]+\frac{dw}{d\theta}\left[\left(\cancel{u}+\frac{dw}{d\theta}\right)\cot\theta+\mu\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)\right]$$

$$\left.+u\left[\cancel{\frac{dw}{d\theta}\cot\theta+\mu\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)}\right]\right\}-\phi\left(u+\frac{dw}{d\theta}\right)$$

$$+\frac{dw}{d\theta}\left\{\cancel{u}\cot\theta-w+\mu\left[\frac{du}{d\theta}-w+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right]\right\}+u\left\{\cancel{-w+\mu\left[-w+\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right]}\right\}=0$$

(135)

$$(1+\alpha)\left[\frac{d^2u}{d\theta^2}+\cot\theta\frac{du}{d\theta}-(\mu+\phi+\cot^2\theta)u\right]=(1+\mu)[1+w]\frac{dw}{d\theta}$$

$$-\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\left\{\mu\frac{dw}{d\theta}+(1-\mu)\cot\theta\right\}\cancel{+\frac{dw}{d\theta}\left[\frac{dw}{d\theta}\cdot\cot\theta+\mu\frac{d^2w}{d\theta^2}\right]}+\phi\left(\frac{dw}{d\theta}\right)$$

$$-\alpha\left\{\frac{d^3w}{d\theta^3}+\cot\theta\frac{d^2w}{d\theta^2}-(\mu+\cot^2\theta)\frac{dw}{d\theta}+\frac{dw}{d\theta}\left(\frac{dw}{d\theta}\cot\theta+\mu\frac{d^2w}{d\theta^2}\right)\right\}$$

or

$$\boxed{\begin{aligned}&(1+\alpha)\left[\frac{d^2u}{d\theta^2}+\cot\theta\cdot\frac{du}{d\theta}-(\mu+\phi+\cot^2\theta)u\right]\\&=(1+\mu)(1+w)\frac{dw}{d\theta}+\phi\left(\frac{dw}{d\theta}\right)-\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\left\{\mu\frac{dw}{d\theta}+(1-\mu)\cot\theta\right\}\\&-\alpha\left\{\frac{d^3w}{d\theta^3}+\cot\theta\frac{d^2w}{d\theta^2}+\left(\frac{dw}{d\theta}\cot\theta+\mu\frac{d^2w}{d\theta^2}-\mu-\cot^2\theta\right)\frac{dw}{d\theta}\right\}\end{aligned}}$$

$$\phi=\frac{p(1-\mu^2)}{2E\left(\frac{t}{r}\right)}$$

But $\quad p\sim\dfrac{E\left(\frac{t}{r}\right)^2}{\sqrt{3(1-\mu^2)}}$

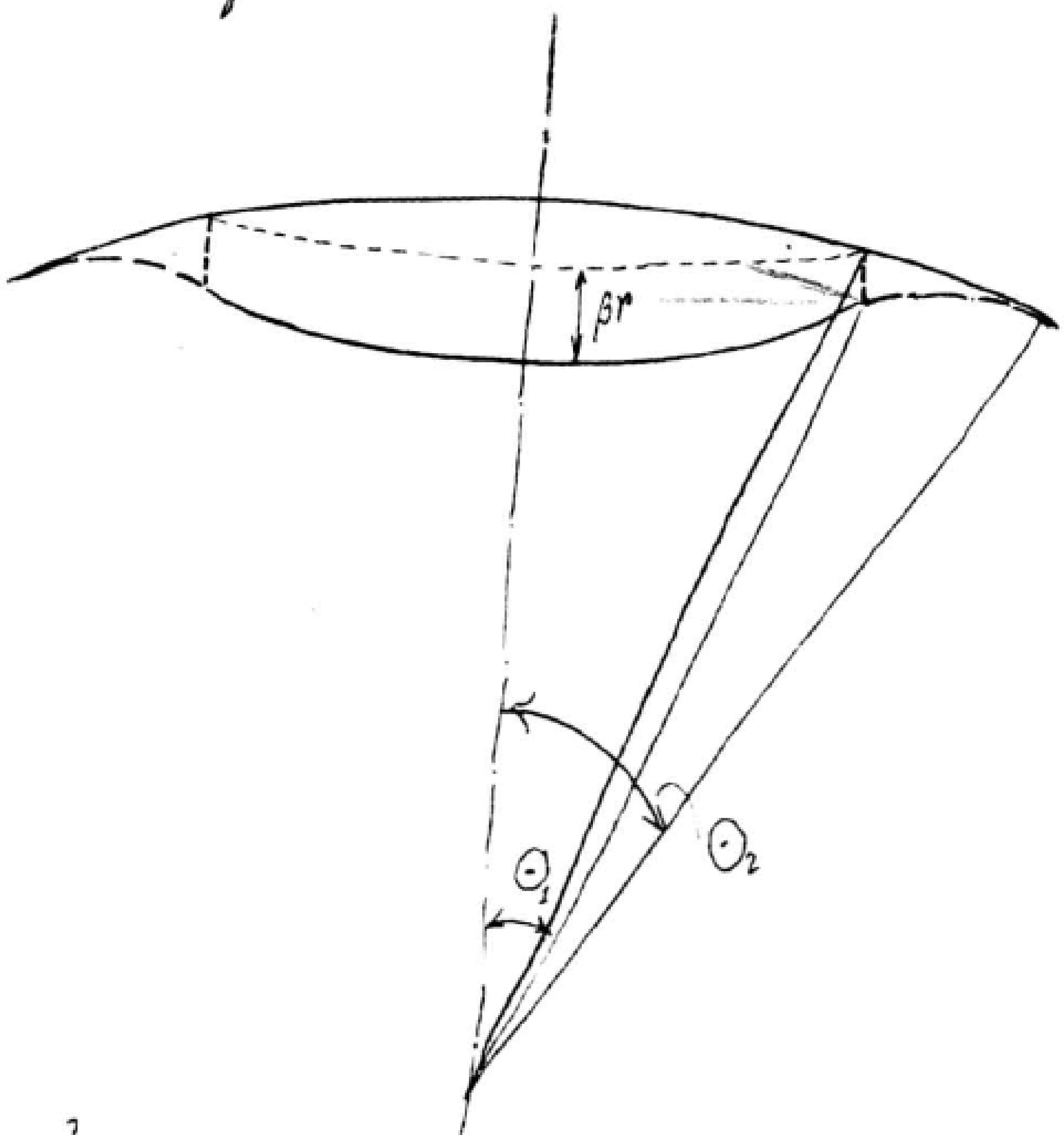
$$\therefore\quad\phi\sim\left(\frac{t}{r}\right)\qquad\qquad\alpha\sim\left(\frac{t}{r}\right)^2,$$

$\therefore$ the differential equation for u can be simplified when θ is small,

$$\boxed{\begin{aligned}&\frac{d^2u}{d\theta^2}+\frac{1}{\theta}\frac{du}{d\theta}-\frac{u}{\theta^2}=(1+\mu)(1+w)\frac{dw}{d\theta}+\phi\left(\frac{dw}{d\theta}\right)-\frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\left\{\mu\frac{dw}{d\theta}+\frac{(1-\mu)}{\theta}\right\}\\&-\alpha\left\{\frac{d^3w}{d\theta^3}+\cancel{\cot\theta}\frac{1}{\theta}\frac{d^2w}{d\theta^2}+\left(\frac{1}{\theta}\frac{dw}{d\theta}+\mu\frac{d^2w}{d\theta^2}\cancel{-\mu}-\frac{1}{\theta^2}\right)\frac{dw}{d\theta}\right\}\end{aligned}}$$

Assume the center part of the shell ~~falling~~ only (136)

reflects itself.



For Θ_1

$$w = -r\left\{1 + 4\left(\cos\theta - \cos\Theta_1 + \frac{\beta}{2}\right)^2 - 4\left(\cos\theta - \cos\Theta_1 + \frac{\beta}{2}\right)\cos\theta\right\}^{\frac{1}{2}} - r$$

$$\approx \underline{\underline{r\beta}}$$

$$\frac{1}{r}\frac{dw}{d\theta} \sim -2\Theta_1$$

$$u \approx r\beta \cdot \Theta_1$$

Let beyond Θ_1, $\quad \frac{w}{r} = k_1(\Theta_2 - \theta)^2 + k_2(\Theta_2 - \theta)^3$

$\therefore \quad \cancel{r\beta = k(\Theta_2 - \Theta_1)^2 \quad 2\Theta_1 = -2k(\Theta_2 - \Theta_1)}$

$$r\beta = rk_1(\Theta_2-\Theta_1)^2 + rk_2(\Theta_2-\Theta_1)^3 \qquad (137)$$

$$2\Theta_1 = +2k_1(\Theta_2-\Theta_1) + 3k_2(\Theta_2-\Theta_1)^2$$

$$\beta = k_1(\Theta_2-\Theta_1)^2 + k_2(\Theta_2-\Theta_1)^3$$

$$k_1 = \frac{2\Theta_1(\Theta_2-\Theta_1)^3 - 3\beta(\Theta_2-\Theta_1)^2}{(\Theta_2-\Theta_1)^4(2-3)}$$

$$= -\frac{2\Theta_1(\Theta_2-\Theta_1) - 3\beta}{(\Theta_2-\Theta_1)^2} = \frac{3\beta - 2\Theta_2^2\xi(1-\xi)}{\Theta_2^2(1-\xi)^2}$$

$$k_2 = -\frac{\beta\, 2(\Theta_2-\Theta_1) - (\Theta_2-\Theta_1)^2\, 2\Theta_1}{(\Theta_2-\Theta_1)^4}$$

$$= -\frac{2\beta - 2\Theta_1(\Theta_2-\Theta_1)}{(\Theta_2-\Theta_1)^3} = \frac{2\Theta_2^2\xi(1-\xi) - 2\beta}{\Theta_2^3(1-\xi)^3}$$

Therefore if we put $(\Theta_2-\Theta_1) = \gamma$

$$\boxed{\bar{w} = \frac{w_1}{r} = \left(\frac{3\beta - 2\Theta_1\gamma}{\gamma^2}\right)(\Theta_2-\theta)^2 + \left(\frac{2\Theta_1\gamma - 2\beta}{\gamma^3}\right)(\Theta_2-\theta)^3}$$

$$\frac{dw}{d\theta} = -2k_1(\Theta_2-\theta) - 3k_2(\Theta_2-\theta)^2$$

$$\frac{d^2w}{d\theta^2} = 2k_1 + 6k_2(\Theta_2-\theta)$$

$$\frac{d^3w}{d\theta^3} = -6k_2$$

The important terms of the right hand side of the diff. eqn. 138) for u is

$$(1+\mu)\frac{dw}{d\theta} - \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\frac{(1-\mu)}{\theta} - \alpha\left\{\frac{d^3w}{d\theta^3} + \frac{1}{\theta}\frac{d^2w}{d\theta^2} - \frac{1}{\theta^2}\frac{dw}{d\theta}\right\}$$

$$= -(1+\mu)\left[2k_1(\Theta_2-\theta) + 3k_2(\Theta_2-\theta)^2\right] - \frac{(1-\mu)}{2\theta}\left[2k_1(\Theta_2-\theta) + 3k_2(\Theta_2-\theta)^2\right]^2$$

$$\underbrace{-\alpha\left\{-6k_2 + \frac{2k_1}{\theta} + 6k_2\left(\frac{\Theta_2}{\theta}-1\right) + \frac{2k_1}{\theta}\left(\frac{\Theta_2}{\theta}-1\right) + 3k_2\left(\frac{\Theta_2}{\theta}-1\right)^2\right\}}_{\text{small}}$$

$$\approx -\left[2(1+\mu)k_1(\Theta_2-\theta) + 3(1+\mu)k_2(\Theta_2-\theta)^2 + \frac{\cancel{4}^{2}(1-\mu)k_1^2}{\cancel{8}\theta}(\Theta_2-\theta)^2 + \frac{9(1-\mu)k_2^2}{2\theta}(\Theta_2-\theta)^4 + \frac{6k_1k_2(1-\mu)}{\theta}(\Theta_2-\theta)^3\right]$$

$$\cancel{= -\left[9(1-\mu)\right.}$$

$$= -\Big[(1-\mu)\left(2k_1^2 + \frac{9}{2}k_2^2\Theta_2^2 + 6k_1k_2\Theta_2\right)\frac{\Theta_2^2}{\theta} + \left\{(1+\mu)(2k_1 + 3k_2\Theta_2) - (1-\mu)(4k_1^2 + 18k_2^2\Theta_2^2 + 18k_1k_2\Theta_2)\right\}\Theta_2$$
$$+ \left\{(1-\mu)(2k_1^2 + 27k_2^2\Theta_2^2 + 18k_1k_2\Theta_2) - (1+\mu)(2k_1 + 6k_2\Theta_2)\right\}\theta$$
$$+ \left\{3(1+\mu)k_2 - (1-\mu)(18k_2^2\Theta_2 + 6k_1k_2)\right\}\theta^2 + \frac{9}{2}(1-\mu)k_2^2\theta^3\Big]$$

$$= -\left[\eta_1\frac{1}{\theta} + \eta_2 + \eta_3\theta + \eta_4\theta^2 + \eta_5\theta^3\right]$$

$$\frac{d^2u}{d\theta^2} + \frac{1}{\theta}\frac{du}{d\theta} - \frac{u}{\theta^2} = -\left[\eta_1 \frac{1}{\theta} + \eta_2 + \eta_3\theta + \eta_4\theta^2 + \eta_5\theta^3\right]$$

$$\theta^2 \frac{d^2u}{d\theta^2} + \theta\frac{du}{d\theta} - u = -\left[\eta_1\theta + \eta_2\theta^2 + \eta_3\theta^3 + \eta_4\theta^4 + \eta_5\theta^5\right]$$

Put $\theta = e^z$

$$(D^2 - D)u + Du - u = -\left[\eta_1 e^z + \eta_2 e^{2z} + \eta_3 e^{3z} + \eta_4 e^{4z} + \eta_5 e^{5z}\right]$$

$$D^2 u - u = -\left[\eta_1 e^z + \eta_2 e^{2z} + \eta_3 e^{3z} + \eta_4 e^{4z} + \eta_5 e^{5z}\right]$$

The complementary function

$$u = Ae^z + Be^{-z}$$

the particular solution

$$Ce^{mz}$$

$$C[m^2-1](e^{mz}) = -\eta_m(e^{mz})$$

$$C = -\frac{\eta_m}{m^2-1}, \quad \text{for} \quad e^z$$

Take $C_1 z e^z$

$$(D^2-1)u = C_1\left\{2e^z\right\} = -\eta_1 e^z \qquad C_1 = -\frac{\eta_1}{2}$$

$$u = Ae^z + Be^{-z} - \frac{\eta_1}{2} z e^z - \frac{\eta_2}{3}e^{2z} - \frac{\eta_3}{8}e^{3z} - \frac{\eta_4}{15}e^{4z} - \frac{\eta_5}{24}e^{5z}$$

$$= A\theta + \frac{B}{\theta} - \frac{\eta_1}{2}\theta\log\theta - \frac{\eta_2}{3}\theta^2 - \frac{\eta_3}{8}\theta^3 - \frac{\eta_4}{15}\theta^4 - \frac{\eta_5}{24}\theta^5$$

To find the value of $\frac{du}{d\theta}$ at Θ_1

we have $\qquad \varepsilon_1 = \frac{du}{d\theta} - w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 = 0$

$$\frac{du}{d\theta} = w - \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2$$

$$w = \frac{\bar{w}}{r} = \left\{1 + 4\left(\cos\theta - \cos\Theta_1 + \frac{\beta}{2}\right)^2 - 4\left(\cos\theta - \cos\Theta_1 + \frac{\beta}{2}\right)\cos\theta\right\}^{\frac{1}{2}} - 1$$

$$w(\Theta_1) = \left[\left\{1 + \beta^2 - 2\beta\cos\Theta_1\right\}^{\frac{1}{2}} - 1\right] \simeq \left[\left\{1 + \beta^2 - 2\beta\right\}^{\frac{1}{2}} - 1\right]$$

$$= +\beta.$$

$$\frac{du}{d\theta} = \beta - \frac{1}{2}(2\Theta_1)^2 = \beta - 2\Theta_1^2$$

$$\beta\Theta_1 = A\Theta_1 + \frac{B}{\Theta_1} - \frac{\eta_1}{2}\Theta_1\log\Theta_1 - \frac{\eta_2}{3}\Theta_1^2 - \frac{\eta_3}{8}\Theta_1^3 - \frac{\eta_4}{15}\Theta_1^4 - \frac{\eta_5}{24}\Theta_1^5$$

$$\beta - 2\Theta_1^2 = A - \frac{B}{\Theta_1^2} - \frac{\eta_1}{2}(1 + \log\Theta_1) - \frac{2}{3}\eta_2\Theta_1 - \frac{3}{8}\eta_3\Theta_1^2 - \frac{4}{15}\eta_4\Theta_1^3$$
$$- \frac{5}{24}\eta_5\Theta_1^4$$

or $\qquad A\Theta_1 + \frac{B}{\Theta_1} = \zeta_1 = \beta\Theta_1 + \frac{\eta_1}{2}\Theta_1\log\Theta_1 + \frac{\eta_2}{3}\Theta_1^2 + \frac{\eta_3}{8}\Theta_1^3 + \frac{\eta_4}{15}\Theta_1^4 + \frac{\eta_5}{24}\Theta_1^5$

$$A\Theta_1 - \frac{B}{\Theta_1} = \zeta_2 = \beta\Theta_1 - 2\Theta_1^3 + \frac{\eta_1}{2}(\Theta_1 + \Theta_1\log\Theta_1) + \frac{2}{3}\eta_2\Theta_1^2$$
$$+ \frac{3}{8}\eta_3\Theta_1^3 + \frac{4}{15}\eta_4\Theta_1^4 + \frac{5}{24}\eta_5\Theta_1^5$$

$$\boxed{A = \frac{1}{2\Theta_1}(\zeta_1 + \zeta_2), \qquad B = \frac{\Theta_1}{2}(\zeta_1 - \zeta_2)}$$

(141)

$$S_1+S_2 = 2\beta\Theta_1 - 2\Theta_1^3 + \frac{\eta_1}{2}(\Theta_1 + 2\Theta_1\log\Theta_1) + \eta_2\Theta_1^2 + \frac{\eta_3}{2}\Theta_1^3 + \frac{\eta_4}{3}\Theta_1^4 + \frac{\eta_5}{4}\Theta_1^5$$

$$\boxed{\begin{aligned} A &= \beta - \Theta_1^2 + \frac{\eta_1}{2}\left(\frac{1}{2}+\log\Theta_1\right) + \frac{\eta_2}{2}\Theta_1 + \frac{\eta_3}{4}\Theta_1^2 + \frac{\eta_4}{6}\Theta_1^3 + \frac{\eta_5}{8}\Theta_1^4 \\ B &= \Theta_1^4 - \frac{\eta_1}{4}\Theta_1^2 - \frac{\eta_2}{6}\Theta_1^3 - \frac{\eta_3}{8}\Theta_1^4 - \frac{\eta_4}{10}\Theta_1^5 - \frac{\eta_5}{12}\Theta_1^6 \end{aligned}}$$

$$\frac{du}{d\theta} - 2w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 + \frac{u}{\theta}$$

$$= A - \frac{B}{\theta^2} - \frac{\eta_1}{2}(1+\log\theta) - \frac{2}{3}\eta_2\theta - \frac{3}{8}\eta_3\theta^2 - \frac{4}{15}\eta_4\theta^3 - \frac{5}{24}\eta_5\theta^4$$

$$- 2k_1(\Theta_2-\theta)^2 - 2k_2(\Theta_2-\theta)^3 + \frac{1}{2}\left[2k_1(\Theta_2-\theta) + 3k_2(\Theta_2-\theta)^2\right]^2$$

$$+ A + \frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta - \frac{\eta_2}{3}\theta - \frac{\eta_3}{8}\theta^2 - \frac{\eta_4}{15}\theta^3 - \frac{\eta_5}{24}\theta^4$$

$$= 2A - \frac{\eta_1}{2}(1+2\log\theta) - \eta_2\theta - \frac{\eta_3}{2}\theta^2 - \frac{\eta_4}{3}\theta^3 - \frac{\eta_5}{4}\theta^4$$

$$+\left[-2k_1\Theta_2^2 - 2k_2\Theta_2^3 + \frac{1}{2}(2k_1\Theta_2 + 3k_2\Theta_2^2)^2\right]$$

$$+\left[4k_1\Theta_2 + 6k_2\Theta_2^2 - 2(2k_1\Theta_2 + 3k_2\Theta_2^2)(k_1+3k_2\Theta_2)\right]\theta$$

$$+\left[-2k_1 - 6k_2\Theta_2 + 2(k_1+3k_2\Theta_2)^2 + 3k_2(2k_1\Theta_2 + 3k_2\Theta_2^2)\right]\theta^2$$

$$+\left[+2k_2 - 6k_2(k_1+3k_2\Theta_2)\right]\theta^3$$

$$+ \frac{9}{2}k_2^2\theta^4$$

$$= \left[2A - 2k_1\Theta_2^2 - 2k_2\Theta_2^3 + \frac{1}{2}(2k_1\Theta_2 + 3k_2\Theta_2^2)^2 - \frac{\eta_1}{2}\right] - \eta_1 \log\theta \quad (1\text{-}2)$$

$$+ \left[4k_1\Theta_2 + 6k_2\Theta_2^2 - 2(2k_1\Theta_2 + 3k_2\Theta_2^2)(k_1 + 3k_2\Theta_2) - \eta_2\right]\theta$$

$$+ \left[-2k_1 - 6k_2\Theta_2 + 2(k_1 + 3k_2\Theta_2)^2 + 3k_2(2k_1\Theta_2 + 3k_2\Theta_2^2) - \frac{\eta_3}{2}\right]\theta^2$$

$$+ \left[+2k_2 - 6k_2(k_1 + 3k_2\Theta_2) - \frac{\eta_4}{3}\right]\theta^3$$

$$+ \left(\frac{9}{2}k_2^2 - \frac{\eta_5}{4}\right)\theta^4$$

$$= C_0 - \eta_1 \log\theta + C_1\theta + C_2\theta^2 + C_3\theta^3 + C_4\theta^4$$

$$\left[\frac{du}{d\theta} - 2w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2 + \frac{u}{\theta}\right]^2$$

$$= C_0^2 - 2\eta_1(C_0 + C_1\theta + C_2\theta^2 + C_3\theta^3 + C_4\theta^4)\log\theta + \eta_1^2(\log\theta)^2$$

$$+ 2C_0C_1\theta + (C_1^2 + 2C_0C_2)\theta^2 + (2C_0C_3 + 2C_1C_2)\theta^3 + (C_2^2 + 2C_0C_4 + 2C_1C_3)\theta^4$$

$$+ (2C_1C_4 + 2C_2C_3)\theta^5 + (C_3^2 + 2C_2C_4)\theta^6 + (2C_3C_4)\theta^7 + C_4^2\theta^8$$

$$\text{(I)} = \frac{C_0^2}{2}\theta^2 + \frac{2}{3}C_0C_1\theta^3 + \boxed{\frac{1}{4}(C_1^2 + 2C_0C_2)\theta^4 + \frac{1}{5}(2C_0C_3 + 2C_1C_2)\theta^5}$$

$$\boxed{+ \frac{1}{6}(C_2^2 + 2C_0C_4 + 2C_1C_3)\theta^6 + \frac{1}{7}(2C_1C_4 + 2C_2C_3)\theta^7 + \frac{1}{8}(C_3^2 + 2C_2C_4)\theta^8}$$

$$+ \boxed{\frac{2}{9}C_3C_4\theta^9 + \frac{C_4^2}{10}\theta^{10}} + \frac{\eta_1^2}{2}\left[(\log\theta)^2 - \log\theta + \frac{1}{2}\right]\theta^2$$

$$-2\eta_1\left\{\frac{C_0}{2}\theta^2\left(\log\theta - \frac{1}{2}\right) + \frac{C_1}{3}\theta^3\left(\log\theta - \frac{1}{3}\right) + \frac{C_2}{4}\theta^4\left(\log\theta - \frac{1}{4}\right) + \boxed{\frac{C_3}{5}\theta^5\left(\log\theta - \frac{1}{5}\right)}\right.$$
$$\left. + \frac{C_4}{6}\theta^6\left(\log\theta - \frac{1}{6}\right)\right\}$$

$$\left[\frac{du}{d\theta} - w + \frac{1}{2}\left(\frac{dw}{d\theta}\right)^2\right]\left[\frac{u}{\theta} - w\right]$$

$$= \Big\{ A - \frac{B}{\theta^2} - \frac{\eta_1}{2}(1 + \log\theta) - \frac{2}{3}\eta_2\theta - \frac{3}{8}\eta_3\theta^2 - \frac{4}{15}\eta_4\theta^3 - \frac{5}{24}\eta_5\theta^4$$
$$- k_1(\Theta_2 - \theta)^2 - k_2(\Theta_2 - \theta)^3 + \frac{1}{2}\left[2k_1(\Theta_2 - \theta) + 3k_2(\Theta_2 - \theta)^2\right]^2 \Big\}$$
$$\Big\{ A + \frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta - \frac{\eta_2}{3}\theta - \frac{\eta_3}{8}\theta^2 - \frac{\eta_4}{15}\theta^3 - \frac{\eta_5}{24}\theta^4 - k_1(\Theta_2 - \theta)^2 - k_2(\Theta_2 - \theta)^3 \Big\}$$

$$= \Big\{ -\frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta + \left[A - \frac{\eta_1}{2} - k_1\Theta_2^2 - k_2\Theta_2^3 + \frac{1}{2}(2k_1\Theta_2 + 3k_2\Theta_2^2)^2\right]$$
$$+ \left[-\frac{2}{3}\eta_2 + 2k_1\Theta_2 + 3k_2\Theta_2^2 - 2(k_1 + 3k_2\Theta_2)(2k_1\Theta_2 + 3k_2\Theta_2^2)\right]\theta$$
$$+ \left[-\frac{3}{8}\eta_3 - k_1 - 3k_2\Theta_2 + 2(k_1 + 3k_2\Theta_2)^2 + 3k_2(2k_1\Theta_2 + 3k_2\Theta_2^2)\right]\theta^2$$
$$+ \left[-\frac{4}{15}\eta_4 + k_2 - 6k_2(k_1 + 3k_2\Theta_2)\right]\theta^3$$
$$+ \left[-\frac{5}{24}\eta_5 + \frac{9}{2}k_2^2\right]\theta^4 \Big\}$$

$$\times \Big\{ + \frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta + \{A - k_1\Theta_2^2 - k_2\Theta_2^3\} + \{-\frac{\eta_2}{3} + 2k_1\Theta_2 + 3k_2\Theta_2^2\}\theta$$
$$+ \{-\frac{\eta_3}{8} - k_1 - 3k_2\Theta_2\}\theta^2 + \{-\frac{\eta_4}{15} + k_2\}\theta^3 - \frac{\eta_5}{24}\theta^4$$

$$= \Big\{ -\frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta + C_5 + C_6\theta + C_7\theta^2 + C_8\theta^3 + C_9\theta^4 \Big\}$$
$$\times \Big\{ \frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta + C_{10} + C_{11}\theta + C_{12}\theta^2 + C_{13}\theta^3 + C_{14}\theta^4 \Big\}$$

$$= \left(\frac{\eta}{2}\right)^2(\log\theta)^2 - \frac{\eta}{2}\log\theta\left[(C_5+C_{10})+(C_6+C_{11})\theta+(C_7+C_{12})\theta^2+(C_8+C_{13})\theta^3+(C_9+C_{14})\theta^4\right]$$

$$-\frac{B^2}{\theta^4} + B(C_5 - C_{10})\frac{1}{\theta^2} + B(C_6 - C_{11})\frac{1}{\theta} + (C_7 B + C_5 C_{10} - C_{12} B)$$

$$+ (BC_8 + C_6 C_{10} + C_5 C_{11} - BC_{13})\theta + (BC_9 + C_7 C_{10} + C_6 C_{11} + C_5 C_{12} - BC_{14})\theta^2$$

$$+ (C_8 C_{10} + C_7 C_{11} + C_6 C_{12} + C_5 C_{13})\theta^3$$

$$+ (C_9 C_{10} + C_8 C_{11} + C_7 C_{12} + C_6 C_{13} + C_5 C_{14})\theta^4$$

$$+ (C_9 C_{11} + C_8 C_{12} + C_7 C_{13} + C_6 C_{14})\theta^5$$

$$+ (C_9 C_{12} + C_8 C_{13} + C_7 C_{14})\theta^6$$

$$+ (C_9 C_{13} + C_8 C_{14})\theta^7$$

$$+ (C_9 C_{14})\theta^8$$

$$\text{II} = \frac{\eta^2}{8}\left\{(\log\theta)^2 - \log\theta + \frac{1}{2}\right\}\theta^2 - \frac{\eta}{2}\left\{\frac{(C_5+C_{10})}{2}\theta^2\left(\log\theta - \frac{1}{2}\right) + \frac{(C_6+C_{11})}{3}\theta^3\left(\log\theta - \frac{1}{3}\right)\right.$$

$$\left. + \frac{(C_7+C_{12})}{4}\theta^4\left(\log\theta - \frac{1}{4}\right) + \frac{(C_8+C_{13})}{5}\theta^5\left(\log\theta - \frac{1}{5}\right) + \frac{(C_9+C_{14})}{6}\theta^6\left(\log\theta - \frac{1}{6}\right)\right\}$$

$$+ \frac{1}{2}\frac{B^2}{\theta^2} + B(C_5 - C_{10})\log\theta + B(C_6 - C_{11})\theta + \frac{BC_7 + C_5 C_{10} - BC_{12}}{2}\theta^2$$

$$+ \frac{1}{3}(BC_8 + C_6 C_{10} + C_5 C_{11} - BC_{13})\theta^3 + \frac{1}{4}(BC_9 + C_7 C_{10} + C_6 C_{11} + C_5 C_{12} - BC_{14})\theta^4$$

$$+ \frac{1}{5}(C_8 C_{10} + C_7 C_{11} + C_6 C_{12} + C_5 C_{13})\theta^5 + \frac{1}{6}(C_9 C_{10} + C_8 C_{11} + C_7 C_{12} + C_6 C_{13} + C_5 C_{14})\theta^6$$

$$+ \frac{1}{7}(C_9 C_{11} + C_8 C_{12} + C_7 C_{13} + C_6 C_{14})\theta^7 + \frac{1}{8}(C_9 C_{12} + C_8 C_{13} + C_7 C_{14})\theta^8$$

$$+ \frac{1}{9}(C_9 C_{13} + C_8 C_{14})\theta^9 + \frac{1}{10}(C_9 C_{14})\theta^{10}.$$

$$\frac{d^2w}{d\theta^2} + \frac{du}{d\theta} + \frac{u}{\theta} + \frac{1}{\theta}\frac{dw}{d\theta}$$

$$= (2k_1 + 6k_2\Theta_2) - 6k_2\theta - \frac{(2k_1\Theta_2 + 3k_2\Theta_2^2)}{\theta} + 2(k_1 + 3k_2\Theta_2) - 3k_2\theta$$

$$+ 2A - \eta_1 \log\theta - \frac{\eta_1}{2} - \eta_2\theta - \frac{\eta_3}{2}\theta^2 - \frac{\eta_4}{3}\theta^3 - \frac{\eta_5}{4}\theta^4$$

$$= -\frac{(2k_1\Theta_2 + 3k_2\Theta_2^2)}{\theta} + \left(2A + 4k_1 + 12k_2\Theta_2 - \frac{\eta_1}{2}\right) + (-9k_2 - \eta_2)\theta$$

$$- \frac{\eta_3}{2}\theta^2 - \frac{\eta_4}{3}\theta^3 - \frac{\eta_5}{4}\theta^4$$

$$= \frac{G_1}{\theta} + G_2 + G_3\theta - \frac{\eta_3}{2}\theta^2 - \frac{\eta_4}{3}\theta^3 - \frac{\eta_5}{4}\theta^4 - \eta_1\log\theta$$

$$\left[\frac{dw}{d\theta^2} + \frac{du}{d\theta} + \frac{u}{\theta} + \frac{1}{\theta}\frac{dw}{d\theta}\right]^2$$

$$= \frac{G_1^2}{\theta^2} + \frac{2G_1G_2}{\theta} + (G_2^2 + 2G_1G_3) + (2G_2G_3 - \eta_3G_1)\theta$$

$$+ (G_3^2 - \eta_4G_1 - \eta_3G_2)\theta^2 - (\eta_4G_2 + \eta_3G_3 + \eta_5G_1)\theta^3$$

$$- \left(\eta_4G_3 + \eta_5G_2 - \frac{\eta_3^2}{4}\right)\theta^4 - \left(\eta_5G_3 - \frac{\eta_3\eta_4}{3}\right)\theta^5 + \left(\frac{\eta_4^2}{9} + \frac{\eta_3\eta_5}{4}\right)\theta^6$$

$$+ \frac{\eta_4\eta_5}{6}\theta^7 + \frac{\eta_5^2}{16}\theta^8$$

$$- 2\eta_1\log\theta\left(\frac{G_1}{\theta} + G_2 + G_3\theta - \frac{\eta_3}{2}\theta^2 - \frac{\eta_4}{3}\theta^3 - \frac{\eta_5}{4}\theta^4\right)$$

$$\text{III} = G_1^2 \log\theta + 2G_1G_2\theta + \frac{(G_2^2+2G_1G_3)}{2}\theta^2 + \frac{(2G_2G_3-\eta_3G_1)}{3}\theta^3 \qquad 146)$$

$$+ \frac{(G_3^2-\eta_4G_1-\eta_3G_2)}{4}\theta^4 - \frac{(\eta_4G_2+\eta_3G_3+\eta_5G_1)}{5}\theta^5 - \frac{(\eta_4G_3+\eta_5G_2-\frac{\eta_3^2}{4})}{6}\theta^6$$

$$- \frac{(\eta_5G_3-\frac{\eta_3\eta_4}{3})}{7}\theta^7 + \frac{(\frac{\eta_4^2}{9}+\frac{\eta_3\eta_5}{4})}{8}\theta^8 + \frac{\eta_4\eta_5}{54}\theta^9 + \frac{\eta_5^2}{160}\theta^{10}$$

$$- 2\eta_1\Big[G_1\theta(\log\theta-1) + \frac{G_2}{2}\theta^2(\log\theta-\tfrac{1}{2}) + \frac{G_3}{3}\theta^3(\log\theta-\tfrac{1}{3})$$

$$- \frac{\eta_3}{8}\theta^4(\log\theta-\tfrac{1}{4}) - \frac{\eta_4}{15}\theta^5(\log\theta-\tfrac{1}{5}) - \frac{\eta_5}{24}\theta^6(\log\theta-\tfrac{1}{6})\Big]$$

$$\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)\left(u+\frac{dw}{d\theta}\right)$$

$$= \Big[(2k_1+6k_2\Theta_2) - 6k_2\theta + \overbrace{}^{(A-\frac{\eta_1}{2})} - \frac{B}{\theta^2} - \frac{\eta_1}{2}\log\theta - \frac{2\eta_2}{3}\theta - \frac{3\eta_3}{8}\theta^2 - \frac{4\eta_4}{15}\theta^3 - \frac{5\eta_5}{24}\theta^4\Big]$$

$$\Big[A\theta + \frac{B}{\theta} - \frac{\eta_1}{2}\theta\log\theta - \frac{\eta_2}{3}\theta^2 - \frac{\eta_3}{8}\theta^3 - \frac{\eta_4}{15}\theta^4 - \frac{\eta_5}{24}\theta^5 - (2k_1\Theta_2+3k_2\Theta_2^2)$$

$$+ 2(k_1+3k_2\Theta_2)\theta - 3k_2\theta^2\Big]$$

$$= \theta\Big[-\frac{B}{\theta^2} + \overbrace{(A-\frac{\eta_1}{2}+2k_1+6k_2\Theta_2)}^{G_4} - \frac{\eta_1}{2}\log\theta - \overbrace{(6k_2+\frac{2\eta_2}{3})}^{G_5}\theta - \frac{3\eta_3}{8}\theta^2 - \frac{4\eta_4}{15}\theta^3 - \frac{5\eta_5}{24}\theta^4\Big]$$

$$\Big[\frac{B}{\theta^2} - \frac{\overbrace{(2k_1\Theta_2+3k_2\Theta_2^2)}^{G_6}}{\theta} + \overbrace{(2k_1+6k_2\Theta_2+A)}^{G_7} - \frac{\eta_1}{2}\log\theta$$

$$- \overbrace{(3k_2+\frac{\eta_2}{3})}^{G_8}\theta - \frac{\eta_3}{8}\theta^2 - \frac{\eta_4}{15}\theta^3 - \frac{\eta_5}{24}\theta^4\Big]$$

continued on p. 148

The strain energy due to extensional deformation is

$$W_{e_c} = \iint \left[\frac{1}{2E}(\sigma_x^2 + \sigma_y^2) - \frac{\mu}{E}\sigma_x\sigma_y\right] dA$$

$$= \frac{1}{E}(1-\mu)\frac{p^2 r^2}{4t^2}\cdot \pi r^2 \Theta_1^2 \; t$$

$$= \frac{(1-\mu)\pi}{4E}\frac{p^2}{\left(\frac{t}{r}\right)} r^3 \Theta_1^2$$

The extension

$$\cancel{\left(\frac{pr}{2t}\right)\cdot t = \frac{Et}{1-\mu^2}\left\{\frac{du}{d\theta} - w + \mu(u\cot\theta - w)\right\}}$$

$$\cancel{= \frac{Et}{1-\mu^2}\left\{u\cot\theta - w + \mu\left(\frac{du}{d\theta} - w\right)\right\}}$$

$$\cancel{\therefore \quad \frac{du}{d\theta} = u\cot\theta \quad \text{or} \quad \frac{du}{d\theta} - \frac{u}{\theta} = 0.}$$

$$\cancel{\text{or} \quad \frac{du}{d\theta} = \frac{u}{\theta} \qquad \frac{du}{u} = \frac{d\theta}{\theta}}$$

$$\cancel{\frac{u}{\theta} = \text{constant}}$$

Consider the whole sphere

$$\frac{1}{E}(1-\mu)\frac{p^2 r^2}{2t^2} 4\pi r^2 t = \frac{1}{2} p \cdot 4\pi r^2 \cdot \delta r$$

$$\delta r = \frac{(1-\mu)}{2E} p \frac{r^2}{t^2} t \qquad\qquad u_{\Theta_1} \approx 0$$

$$\frac{1}{2}\cdot u_{\Theta_1}\cdot 2\pi r\Theta_1 \cdot \frac{pr}{2t}\cdot t + \frac{(1-\mu)}{2E} p \frac{r^2}{t^2}\cdot \frac{1}{2} t \; p \cdot \pi r^2 \Theta_1^2 = \frac{(1-\mu)\pi}{4E}\frac{p^2}{\left(\frac{t}{r}\right)} r^3 \Theta_1^2$$

(p.148)

$$\left(\frac{d^2w}{d\theta^2}+\frac{du}{d\theta}\right)\left(u+\frac{dw}{d\theta}\right)$$

$$= \theta\Big[-\frac{B^2}{\theta^4}+\frac{BG_6}{\theta^3}+\frac{B(G_4-G_7)}{\theta^2}-\frac{(G_5B+G_4G_6-BG_8)}{\theta}+\left(G_4G_7-\frac{\eta_3 B}{4}+G_5G_6\right)$$

$$+\left(\frac{3\eta_3}{8}G_6-G_5G_7-G_8G_4-\frac{\eta_4}{5}B\right)\theta+\left(G_5G_8-\frac{\eta_5}{6}B-\frac{4\eta_4}{15}G_6-\frac{\eta_3}{8}[3G_7+G_4]\right)\theta^2$$

$$+\left[-\frac{5\eta_5}{24}G_6-\frac{\eta_4}{15}(4G_7+G_4)+\frac{\eta_3}{8}(3G_8+G_5)\right]\theta^3$$

$$+\left[-\frac{\eta_5}{24}(5G_7+G_4)+\frac{\eta_4}{15}(4G_8+G_5)+\frac{3\eta_3^2}{64}\right]\theta^4$$

$$+\left[\frac{\eta_5}{24}(5G_8+G_5)+\frac{7\eta_3\eta_4}{120}\right]\theta^5$$

$$+\left[\frac{\eta_3\eta_5}{24}+\frac{4\eta_4^2}{225}\right]\theta^6+\left[\frac{\eta_4\eta_5}{40}\right]\theta^7+\frac{5\eta_5^2}{576}\theta^8+\left(\frac{\eta_1}{2}\right)^2(\log\theta)^2$$

$$+\log\theta\left\{\frac{\eta_1}{2}G_6\frac{1}{\theta}-\frac{\eta_1}{2}(G_4+G_7)+\frac{\eta_1}{2}(G_5+G_8)\theta+\frac{\eta_1\eta_3}{4}\theta^2+\frac{\eta_1\eta_4}{6}\theta^3+\frac{\eta_1\eta_5}{8}\theta^4\right\}\Big]$$

$$\text{IV}=\frac{1}{2}\frac{B^2}{\theta^2}-\frac{BG_6}{\theta}+B(G_4-G_7)\log\theta-(G_5B+G_4G_6-BG_8)\theta$$

$$+\frac{\left(G_4G_7-\frac{\eta_3B}{4}+G_5G_6\right)}{2}\theta^2+\frac{\left(\frac{3\eta_3}{8}G_6-G_5G_7-G_8G_4-\frac{\eta_4}{5}B\right)}{3}\theta^3$$

$$+\frac{\left\{G_5G_8-\frac{\eta_5}{6}B-\frac{4\eta_4}{15}G_6-\frac{\eta_3}{8}(3G_7+G_4)\right\}}{4}\theta^4$$

$$+\frac{\left\{\frac{\eta_3}{8}(3G_8+G_5)-\frac{5\eta_5}{24}G_6-\frac{\eta_4}{15}(4G_7+G_4)\right\}}{5}\theta^5$$

$$+\frac{\left\{\frac{\eta_4}{15}(4G_8+G_5)+\frac{3\eta_3^2}{64}-\frac{\eta_5}{24}(5G_7+G_4)\right\}}{6}\theta^6$$

Contd

$$+\frac{\left\{\frac{\eta_5}{24}(5G_8+G_5)+\frac{7\eta_3\eta_4}{120}\right\}}{7}\theta^7+\frac{1}{8}\left\{\frac{\eta_3\eta_5}{24}+\frac{4\eta_4^2}{225}\right\}\theta^8 \qquad (149)$$

$$+\frac{1}{9}\frac{\eta_4\eta_5}{40}\theta^9+\frac{\eta_5^2}{1152}\theta^{10}+\left(\frac{\eta_1}{2}\right)^2\frac{\theta^2}{2}\left[(\log\theta)^2-\log\theta+\frac{1}{2}\right]$$

$$+\frac{\eta_1}{2}G_6\theta(\log\theta-1)-\frac{\eta_1}{4}(G_4+G_7)\theta^2\left(\log\theta-\frac{1}{2}\right)+\frac{\eta_1}{6}(G_5+G_8)\theta^3\left(\log\theta-\frac{1}{3}\right)$$

$$+\frac{\eta_1\eta_3}{16}\theta^4\left(\log\theta-\frac{1}{4}\right)+\frac{\eta_1\eta_4}{30}\theta^5\left(\log\theta-\frac{1}{5}\right)+\frac{\eta_1\eta_5}{48}\theta^6\left(\log\theta-\frac{1}{6}\right)$$

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for calculating the potential energy.

$$\sin\theta\left[\left(1-\frac{u}{r}\right)^3\left(1+\frac{1}{r}\frac{du}{d\theta}\right)\left(1+\frac{u}{r}\cot\theta\right)\right]$$

$$=\sin\theta\left[\left(1-\frac{3u}{r}+3\left(\frac{u}{r}\right)^2\right)\left(1+\frac{1}{r}\frac{du}{d\theta}\right)\left(1+\frac{u}{r}\cot\theta\right)\right]$$

$$=\sin\theta\left[\left(1-\frac{3u}{r}+3\left(\frac{u}{r}\right)^2\right)\left(1+\frac{1}{r}\frac{du}{d\theta}+\frac{u}{r}\cot\theta+\frac{1}{r}\frac{du}{d\theta}\cdot\frac{u}{r}\cot\theta\right)\right]$$

$$=\sin\theta\left[1+\frac{1}{r}\frac{du}{d\theta}+\frac{u}{r}\cot\theta+\frac{1}{r}\frac{du}{d\theta}\cdot\frac{u}{r}\cot\theta+3\left(\frac{u}{r}\right)^2\right.$$
$$\left.-3\left(\frac{u}{r}\right)\left(1+\frac{1}{r}\frac{du}{d\theta}+\frac{u}{r}\cot\theta\right)\right]$$

The effective terms is

$$\sin\theta\left[\underbrace{\left(1-\frac{3u}{r}\right)\left(\frac{1}{r}\frac{du}{d\theta}+\frac{u}{r}\cot\theta\right)}_{A}\underbrace{-\left(\frac{3u}{r}\right)+\overbrace{\frac{u}{r}\cot\theta\cdot\frac{du}{r\,d\theta}+3\left(\frac{u}{r}\right)^2}^{C}}_{B}\right]$$

$$(1-3w)\left(\frac{du}{d\theta}+\frac{u}{\theta}\right)$$

$$=\Big\{\overbrace{(1-3k_1\Theta_2^2-3k_2\Theta_2^3)}^{F_1}+\overbrace{3(2k_1\Theta_2+3k_2\Theta_2^2)}^{F_2}\theta-\overbrace{3(k_1+3k_2\Theta_2)}^{F_3}\theta^2+3k_2\theta^3\Big\}$$

$$\Big\{\overbrace{\left(2A-\frac{\eta_1}{2}\right)}^{F_4}-\eta_1\log\theta-\eta_2\theta-\frac{\eta_3}{2}\theta^2-\frac{\eta_4}{3}\theta^3-\frac{\eta_5}{4}\theta^4\Big\}$$

$$=\overbrace{\left(2A-\frac{\eta_1}{2}\right)}^{F_4}\overbrace{(1-3k_1\Theta_2^2-3k_2\Theta_2^3)}^{F_1}-\eta_1\log\theta\Big\{\overbrace{(1-3k_1\Theta_2^2-3k_2\Theta_2^3)}^{F_1}+\overbrace{3(2k_1\Theta_2+3k_2\Theta_2^2)}^{F_2}\theta$$
$$-\overbrace{3(k_1+3k_2\Theta_2)}^{F_3}\theta^2+3k_2\theta^3\Big\}$$

$$+\left(F_2F_4-\eta_2F_1\right)\theta-\left(F_4F_3+\frac{\eta_3}{2}F_1+\eta_2F_2\right)\theta^2$$

$$+\left(3k_2F_4-\frac{\eta_4}{3}F_1+\eta_2F_3-\frac{\eta_3}{2}F_2\right)\theta^3-\left(3k_2\eta_2-\frac{\eta_3}{2}F_3+\frac{\eta_5}{4}F_1+\frac{\eta_4}{3}F_2\right)\theta^4$$

$$-\left(\frac{3}{2}\eta_3k_2-\frac{\eta_4}{3}F_3+\frac{\eta_5}{4}F_2\right)\theta^5-\left(k_2\eta_4-\frac{\eta_5}{4}F_3\right)\theta^6-\frac{3}{4}k_2\eta_5\theta^7$$

$$(A)=\frac{F_1F_4}{2}\theta^2+\frac{F_2F_4-\eta_2F_1}{3}\theta^3-\frac{1}{4}\left(F_4F_3+\frac{\eta_3}{2}F_1+\eta_2F_2\right)\theta^4$$

$$+\frac{1}{5}\left(3k_2F_4-\frac{\eta_4}{3}F_1+\eta_2F_3-\frac{\eta_3}{2}F_2\right)\theta^5-\frac{1}{6}\left(3k_2\eta_2-\frac{\eta_3}{2}F_3+\frac{\eta_5}{4}F_1+\frac{\eta_4}{3}F_2\right)\theta^6$$

$$-\frac{1}{7}\left(\frac{3}{2}\eta_3k_2-\frac{\eta_4}{3}F_3+\frac{\eta_5}{4}F_2\right)\theta^7-\frac{1}{8}\left(k_2\eta_4-\frac{\eta_5}{4}F_3\right)\theta^8-\frac{3}{36}k_2\eta_5\theta^9$$

$$-\eta_1\Big\{\frac{F_1}{2}\theta^2\left(\log\theta-\frac{1}{2}\right)+\frac{F_2}{3}\theta^3\left(\log\theta-\frac{1}{3}\right)-\frac{F_3}{4}\theta^4\left(\log\theta-\frac{1}{4}\right)+\frac{3}{5}k_2\theta^5\left(\log\theta-\frac{1}{5}\right)\Big\}$$

151)

$$3\left(\frac{w}{T}\right)\left(\frac{w}{T}-1\right)$$

$$= \left\{ \overbrace{(3k_1\Theta_2^2 + 3k_2\Theta_2^3)}^{3F_5} - 3\overbrace{(2k_1\Theta_2 + 3k_2\Theta_2^2)}^{3F_6}\theta + 3\overbrace{(k_1 + 3k_2\Theta_2)}^{3F_7}\theta^2 - 3k_2\theta^3 \right\}$$

$$\left\{ (k_1\Theta_2^2 + k_2\Theta_2^3 - 1) - (2k_1\Theta_2 + 3k_2\Theta_2^2)\theta + (k_1 + 3k_2\Theta_2)\theta^2 - k_2\theta^3 \right\}$$

$$= 3(F_5 - F_6\theta + F_7\theta^2 - k_2\theta^3)\left((F_5 - 1) - F_6\theta + F_7\theta^2 - k_2\theta^3\right)$$

$$= 3\Big\{ F_5(F_5-1) - (2F_5-1)F_6\theta + [F_6^2 - (2F_5-1)F_7]\theta^2$$

$$- [2F_6F_7 + (2F_5-1)k_2]\theta^3 + (F_7^2 + 2F_6k_2)\theta^4$$

$$- 2k_2F_7\theta^5 + k_2^2\theta^6 \Big\}$$

$$(B) = 3\Big\{ \frac{1}{2}F_5(F_5-1)\theta^2 - \frac{1}{3}F_6(2F_5-1)\theta^3 + \frac{1}{4}[F_6^2 - (2F_5-1)F_7]\theta^4$$

$$- \frac{1}{5}[2F_6F_7 + k_2(2F_5-1)]\theta^5 + \frac{1}{6}(F_7^2 + 2F_6k_2)\theta^6 - \frac{2}{7}k_2F_7\theta^7$$

$$+ \frac{1}{8}k_2^2\theta^8 \Big\}$$

$$\frac{du}{d\theta}\frac{u}{\theta} = \Big\{ -\frac{B}{\theta^2} + \Big(A - \frac{\eta_1}{2}\Big) \boxed{-\frac{\eta_1}{2}\log\theta} - \frac{2}{3}\eta_2\theta - \frac{3}{8}\eta_3\theta^2 - \frac{4}{15}\eta_4\theta^3 - \frac{5}{24}\eta_5\theta^4 \Big\} \quad (159)$$

$$\Big\{ \frac{B}{\theta^2} + A \boxed{-\frac{\eta_1}{2}\log\theta} - \frac{1}{3}\eta_2\theta - \frac{1}{8}\eta_3\theta^2 - \frac{1}{15}\eta_4\theta^3 - \frac{1}{24}\eta_5\theta^4 \Big\}$$

$$= -\frac{B^2}{\theta^4} - \frac{B\eta_1}{2}\frac{1}{\theta^2} - \frac{\eta_1}{2}\log\theta\Big\{\Big(2A - \frac{\eta_1}{2}\Big) - \eta_2\theta - \frac{1}{2}\eta_3\theta^2 - \frac{1}{3}\eta_4\theta^3 - \frac{1}{4}\eta_5\theta^4\Big\}$$

$$-\frac{1}{3}\eta_2 B\frac{1}{\theta} + \Big[A\Big(A - \frac{\eta_1}{2}\Big) - \frac{1}{4}\eta_3 B\Big] - \Big[\frac{1}{5}\eta_4 B + \Big(A - \frac{\eta_1}{6}\Big)\eta_2\Big]\theta$$

$$-\Big[\frac{1}{6}\eta_5 B + \Big(\frac{A}{2} - \frac{\eta_1}{16}\Big)\eta_3 - \frac{2\eta_2}{9}\Big]\theta^2 - \Big[\Big(\frac{A}{3} - \frac{\eta_1}{30}\Big)\eta_4 - \frac{5}{24}\eta_2\eta_3\Big]\theta^3$$

$$-\Big[\Big(\frac{A}{4} - \frac{\eta_1}{48}\Big)\eta_5 - \frac{2}{15}\eta_2\eta_4 - \frac{3}{64}\eta_3^2\Big]\theta^4 + \Big[\frac{7}{72}\eta_2\eta_5 + \frac{7}{120}\eta_3\eta_4\Big]\theta^5$$

$$+\Big[\frac{1}{24}\eta_3\eta_5 + \frac{4}{225}\eta_4^2\Big]\theta^6 + \frac{1}{40}\eta_4\eta_5\,\theta^7 + \frac{5}{576}\eta_5^2\theta^8$$

$$C = \frac{B^2}{2\theta^2} - \frac{B\eta_1}{2}\log\theta - \frac{\eta_1}{2}\Big\{\frac{\big(2A - \frac{\eta_1}{2}\big)}{2}\theta^2\Big(\log\theta - \frac{1}{2}\Big)$$

$$-\frac{\eta_2}{3}\theta^3\Big(\log\theta - \frac{1}{3}\Big) - \frac{1}{8}\eta_3\theta^4\Big(\log\theta - \frac{1}{4}\Big) - \frac{1}{15}\eta_4\theta^5\Big(\log\theta - \frac{1}{5}\Big) - \frac{1}{24}\eta_5\theta^6\Big(\log\theta - \frac{1}{6}\Big)\Big\}$$

$$-\frac{1}{3}\eta_2 B\theta + \frac{1}{2}\Big[A\Big(A - \frac{\eta_1}{2}\Big) - \frac{1}{4}\eta_3 B\Big]\theta^2 - \frac{1}{3}\Big[\frac{1}{5}\eta_4 B + \Big(A - \frac{\eta_1}{6}\Big)\eta_2\Big]\theta^3$$

$$-\frac{1}{4}\Big[\frac{1}{6}\eta_5 B + \Big(\frac{A}{2} - \frac{\eta_1}{16}\Big)\eta_3 - \frac{2\eta_2}{9}\Big]\theta^4 - \frac{1}{5}\Big[\Big(\frac{A}{3} - \frac{\eta_1}{30}\Big)\eta_4 - \frac{5}{24}\eta_2\eta_3\Big]\theta^5$$

$$-\frac{1}{6}\Big[\Big(\frac{A}{4} - \frac{\eta_1}{48}\Big)\eta_5 - \frac{2}{15}\eta_2\eta_4 - \frac{3}{64}\eta_3^2\Big]\theta^6 + \Big[\frac{1}{72}\eta_2\eta_5 + \frac{1}{120}\eta_3\eta_4\Big]\theta^7$$

$$+\frac{1}{8}\Big(\frac{1}{24}\eta_3\eta_5 + \frac{4}{225}\eta_4^2\Big)\theta^8 + \frac{1}{360}\eta_4\eta_5\theta^9 + \frac{1}{1152}\eta_5^2\,\theta^{10}$$

$$k_1 = \frac{3\beta - 2\Theta_2^2\xi(1-\xi)}{\Theta_2^2(1-\xi)^2} = \frac{3\beta}{\Theta_2^2(1-\xi)^2} - 2\frac{\xi}{1-\xi}$$

$1-\xi = 0.4$

Let us put $\frac{\beta}{\Theta_2^2} = \gamma$, now let $\xi = \frac{\Theta_1}{\Theta_2} = 0.6$

$$\boxed{k_1 = \gamma\frac{3}{(1-\xi)^2} - \frac{2\xi}{1-\xi} = \gamma\frac{3}{0.16} - 3 = 18.75\gamma - 3}$$

$$\boxed{k_2 = \frac{1}{\Theta_2}\left\{\frac{2\xi}{(1-\xi)^2} - \frac{2\gamma}{(1-\xi)^3}\right\} = \frac{1}{\Theta_2}\left\{\frac{1.2}{0.16} - \gamma\frac{2}{0.064}\right\} = \frac{1}{\Theta_2}\left\{7.500 - 31.25\gamma\right\}}$$

$$\eta_1 = (1-\mu)\Theta_2^2\left\{2(18.75\gamma-3)^2 + \frac{9}{2}(7.500-31.25\gamma)^2 + 6(18.75\gamma-3)(7.500-31.25\gamma)\right\}$$

$$= (1-\mu)\Theta_2^2\left\{2\left(18.75 - \frac{3}{2}\times 31.25\right)^2\gamma^2 + (33\times 18.75 - 49.5\times 31.25)\gamma + 18 + \frac{7.5^2\times 9}{2} - 13.5\right\}$$

$$\boxed{\eta_1 = (1-\mu)\Theta_2^2\left\{1582.03125\gamma^2 - 928.125\gamma + 136.125\right\}}$$

$$\frac{\eta_2}{\Theta_2} = (1+\mu)(37.5\gamma - 6 + 22.50 - 93.75\gamma) - (1-\mu)\Big[4(18.75\gamma-3)^2 + 18(7.500-31.25\gamma)^2 + 18(18.75\gamma-3)(7.500-31.25\gamma)\Big]$$

$$= (1+\mu)(16.50 - 56.25\gamma) - (1-\mu)\Big[(4\times 18.75^2 + 18\times 31.25^2 - 18\times 18.75\times 31.25)\gamma^2 - (24\times 18.75 + 36\times 7.500\times 31.25 - 18\times 7.500\times 18.75 - 54\times 31.25)\gamma + 36 + 18\times 7.500^2 - 54\times 7.500\Big]$$

$$\boxed{\frac{\eta_2}{\Theta_2} = (1+\mu)(16.50 - 56.25\gamma) - (1-\mu)(8437.5\gamma^2 - 4668.75\gamma + 643.5)}$$

$$\eta_3 = (1-\mu)\left[2(18.75\gamma-3)^2+27(7.500-31.25\gamma)^2+18(18.75\gamma-3)(7.500-31.25\gamma)\right]$$

$$-(1+\mu)\left[37.5\gamma-6+45.00-187.5\gamma\right]$$

$$= (1-\mu)\Big[(2\times 18.75^2+27\times 31.25^2-18\times 18.75\times 31.25)\gamma^2$$

$$-(12\times 18.75+54\times 7.500\times 31.25-18\times 18.25\times 7.500-54\times 31.25)\gamma$$

$$+18+27\times 7.500^2-54\times 7.500\Big]$$

$$-(1+\mu)\left[39.00-150\gamma\right]$$

$$\boxed{\eta_3 = (1-\mu)\left[15523.4375\gamma^2-8662.5\gamma+1131.75\right]+(1+\mu)\left[150\gamma-39.00\right]}$$

$$\Theta_2\eta_4 = 3(1+\mu)(7.500-31.25\gamma)-(1-\mu)6\left\{3(7.500-31.25\gamma)+(7.500-31.25\gamma)(18.75\gamma-3)\right\}$$

$$= 3(1+\mu)(7.500-31.25\gamma)-6(1-\mu)\left\{18.75\gamma(7.500-31.25\gamma)\right\}$$

$$\boxed{\Theta_2\eta_4 = (1+\mu)(22.50-93.75\gamma)-(1-\mu)(843.75\gamma-3515.625\gamma^2)}$$

$$\Theta_2 = 0.1 \text{ Radians.}$$

$$\Theta_1 = 0.06 \text{ Radians}$$

$$\underline{\log_e \Theta_1 = -2.81341}$$

$$\Theta_2^2\eta_5 = \frac{9}{2}(1-\mu)(7.500-31.25\gamma)^2 \qquad 155)$$

$$= 4.5(1-\mu)(56.25-468.75\gamma+976.5625\gamma^2)$$

$$\boxed{\Theta_2^2\eta_5 = (1-\mu)(253.125-2109.375\gamma+4394.53125\gamma^2)}$$

$$\frac{A}{\Theta_2^2} = \gamma - \zeta^2 + \left(\frac{1}{4}+\frac{1}{2}\log\Theta_1\right)(1-\mu)(1582.03125\gamma^2 - 928.125\gamma + 136.125)$$

$$+(1+\mu)(8.25-28.125\gamma)\zeta - (1-\mu)(4218.75\gamma^2 - 2334.375\gamma + 321.25)\zeta$$

$$+(1-\mu)\zeta^2[4130.8594\gamma^2 - 2165.625\gamma + 282.9375] + (1+\mu)\zeta^2[37.5\gamma - 9.75]$$

$$+(1+\mu)\zeta^3[3.75-15.625\gamma] - (1-\mu)\zeta^3[140.625\gamma - 585.9375\gamma^2]$$

$$+(1-\mu)\zeta^4[31.640625 - 263.671875\gamma + 549.31640625\gamma^2]$$

$$=(\gamma-0.36) + (1-\mu)(-1829.95\gamma^2 + 1073.57\gamma - 157.457)$$

$$+(1+\mu)(4.95-16.875\gamma) - (1-\mu)(2531.25\gamma^2 - 1400.625\gamma + 193.05)$$

$$+(1-\mu)(1487.11\gamma^2 - 779.625\gamma + 101.858) + (1+\mu)(13.50\gamma - 3.510)$$

$$+(1+\mu)(0.81-3.375\gamma) - (1-\mu)(30.375\gamma - 126.5625\gamma^2)$$

$$+(1-\mu)(4.100625 - 34.1719\gamma + 71.1914\gamma^2)$$

$$\boxed{\frac{A}{\Theta_2^2} = (\gamma-0.36) + (1+\mu)(2.25-6.75\gamma) - (1-\mu)(2676.34\gamma^2 - 1630.02\gamma + 244.548)}$$

156)

$$\frac{B}{\Theta_2^4} = \xi^4 - (1-\mu)\xi^2(395.508\gamma^2 - 232.031\gamma + 34.0313)$$

$$-(1+\mu)\xi^3(2.75 - 9.375\gamma) + (1-\mu)\xi^3(1406.25\gamma^2 - 778.125\gamma + 107.250)$$

$$-(1-\mu)\xi^4(2165.43\gamma^2 - 1062.81\gamma + 141.469) - (1+\mu)\xi^4(18.75\gamma - 4.875)$$

$$-(1+\mu)\xi^5(2.25 - 9.375\gamma) + (1-\mu)\xi^5(84.375\gamma - 351.5625\gamma^2)$$

$$-(1-\mu)\xi^6(21.0938 - 175.781\gamma + 366.211\gamma^2)$$

$$= 0.1296 - (1-\mu)(142.383\gamma^2 - 83.5312\gamma + 12.2513)$$

$$-(1+\mu)(0.594 - 2.025\gamma) + (1-\mu)(303.750\gamma^2 - 168.075\gamma + 23.166)$$

$$-(1-\mu)(280.640\gamma^2 - 140.332\gamma + 18.3344) - (1+\mu)(2.430\gamma - 0.6318)$$

$$-(1+\mu)(0.17496 - 0.729\gamma) + (1-\mu)(6.561\gamma - 27.3375\gamma^2)$$

$$-(1-\mu)(0.984152 - 8.20124\gamma + 17.0859\gamma^2)$$

$$\boxed{\begin{aligned}\frac{B}{\Theta_2^4} &= 0.1296 - (1-\mu)(163.697\gamma^2 - 70.550\gamma + 8.4039)\\ &\quad -(1+\mu)(0.13716 - 0.324\gamma)\end{aligned}}$$

157)

$$\gamma \quad \mu = 0.3$$

$$\frac{\eta_1}{\Theta_2^2} = (1107.72\gamma^2 - 649.688\gamma + 95.2875)$$

$$\frac{\eta_2}{\Theta_2} = -(5906.25\gamma^2 - 3195.00\gamma + 429)$$

$$\eta_3 = (11566.41\gamma^2 - 5866.25\gamma + 741.525)$$

$$\Theta_2\eta_4 = (2460.94\gamma^2 - 712.500\gamma + 29.25)$$

$$\Theta_2^2\eta_5 = (3076.17\gamma^2 - 1147.656\gamma + 177.188)$$

$$\frac{A}{\Theta_2^2} = -(1873.44\gamma^2 - 1133.23\gamma + 142.394)$$

$$\frac{B}{\Theta_2^4} = -(114.588\gamma^2 - 49.606\gamma + 5.83144)$$

$$k_1 = 18.75\gamma - 3$$

$$k_2 = \frac{1}{\Theta_2}[2.500 - 31.25\gamma]$$

158)

$$C_0 = 2A - 2k_1\Theta_2^2 - 2k_2\Theta_2^3 + \frac{1}{2}(2k_1\Theta_2 + 3k_2\Theta_2^2)^2 - \frac{\gamma}{2}$$

$$= -(3746.88\gamma^2 - 2266.46\gamma + 284.588) - 37.50\gamma + 6 - 15.00 + 62.50\gamma$$

$$+ \frac{1}{2}(16.50 - 56.25\gamma)^2 - (553.71\gamma^2 - 324.844\gamma + 47.64375)$$

$$= -(3746.88\gamma^2 - 2266.46\gamma + 284.588) + 25.00\gamma - 9.00$$

$$+ (1582.03125\gamma^2 - 928.125\gamma + 136.125) - (553.71\gamma^2 - 324.844\gamma + 47.64375)$$

$$\boxed{C_0 = -(2718.56\gamma^2 - 1688.17\gamma + 205.107)\Theta_2^2}$$

$$C_1 = 4k_1\Theta_2 + 6k_2\Theta_2^2 - 2(2k_1\Theta_2 + 3k_2\Theta_2^2)(k_1 + 3k_2\Theta_2) - \gamma_2$$

$$= 2(16.50 - 56.25\gamma)[1 - 18.75\gamma + 3 - 22.50 + 93.25\gamma] - \gamma_2$$

$$= (16.50 - 56.25\gamma)(150\gamma - 37) + (5906.25\gamma^2 - 3195.00\gamma + 429)$$

$$= (5906.25\gamma^2 - 3195.00\gamma + 429.000) - (8437.5\gamma^2 - 4556.25\gamma + 610.5)$$

$$\boxed{C_1 = -(2531.25\gamma^2 - 1361.25\gamma + 181.50)\Theta_2}$$

$$\log\Theta_2 = \log 0.1 = -2.302585$$

$$\log\xi = \log 0.6 = -0.510825$$

$$C_2 = 2(k_1 + 3k_2\Theta_2)\left[k_1 + 3k_2\Theta_2 - 1\right] + 3k_2(2k_1\Theta_2 + 3k_2\Theta_2^2) - \frac{\eta_3}{2} \qquad 159)$$

$$= (75.00\gamma - 19.50)(150\gamma - 37) + (22.50 - 93.75\gamma)(16.50 - 56.25\gamma) - \frac{\eta_3}{2}$$

$$= (11250\gamma^2 - 5700\gamma + 721.5) + (5273.4375\gamma^2 - 2812.5\gamma + 371.25)$$

$$- (5783.21\gamma^2 - 2934.375\gamma + 370.7625)$$

$$\boxed{C_2 = (10740.23\gamma^2 - 5578.16\gamma + 721.987)}$$

$$C_3 = 2k_2\left[1 - 3(k_1 + 3k_2\Theta_2)\right] - \frac{\eta_4}{3}$$

$$= -(15.00 - 62.50\gamma)(57.50 - 225\gamma) - (820.313\gamma^2 - 237.5\gamma + 9.91667)$$

$$= -(14062.5\gamma^2 - 6968.75\gamma + 862.5) - (820.313\gamma^2 - 237.5\gamma + 9.91667)$$

$$\boxed{\Theta_2 C_3 = -(14882.8\gamma^2 - 7206.25\gamma + 872.417)}$$

$$C_4 = \frac{1}{2}\left[(3k_2)^2 - \frac{\eta_5}{2}\right]$$

$$= \frac{1}{2}\left[(22.500 - 93.75\gamma)^2 - \frac{\eta_5}{2}\right]$$

$$= \frac{1}{2}\left[(506.25 - 4218.75\gamma + 8789.0625\gamma^2) - (1538.09\gamma^2 - 73.828\gamma + 88.594)\right]$$

$$\boxed{\Theta_2^2 C_4 = (3625.49\gamma^2 - 2072.46\gamma + 208.828)}$$

160)

$$① = \frac{\Theta_2^2}{2}(1-\xi^2)\left\{C_0^2+\frac{1}{2}\eta_1^2+\eta_1 C_0\right\}+\frac{\Theta_2^3}{3}(1-\xi^3)\left\{2C_0C_1+\frac{2}{3}C_1\eta_1\right\}$$

$$+\frac{\Theta_2^4}{4}(1-\xi^4)\left\{C_1^2+2C_0C_2+\frac{1}{2}C_2\eta_1\right\}+\frac{\Theta_2^5}{5}(1-\xi^5)\left\{2C_0C_3+2C_1C_2+\frac{2}{5}C_3\eta_1\right\}$$

$$+\frac{\Theta_2^6}{6}(1-\xi^6)\left\{C_2^2+2C_0C_4+2C_1C_3+\frac{1}{3}C_4\eta_1\right\}+\frac{\Theta_2^7}{7}(1-\xi^7)\left\{2C_1C_4+2C_2C_3\right\}$$

$$+\frac{\Theta_2^8}{8}(1-\xi^8)\left\{C_3^2+2C_2C_4\right\}+\frac{\Theta_2^9}{9}(1-\xi^9)\left\{2C_3C_4\right\}+\frac{\Theta_2^{10}}{10}(1-\xi^{10})\left\{C_4^2\right\}$$

$$-2\eta_1\left[\left(\frac{C_0}{2}+\frac{\eta_1}{4}\right)\Theta_2^2\left\{(1-\xi^2)\log\Theta_2-\xi^2\log\xi\right\}+\frac{C_1}{3}\Theta_2^3\left\{(1-\xi^3)\log\Theta_2-\xi^3\log\xi\right\}\right.$$

$$+\frac{C_2}{4}\Theta_2^4\left\{(1-\xi^4)\log\Theta_2-\xi^4\log\xi\right\}+\frac{C_3}{5}\Theta_2^5\left\{(1-\xi^5)\log\Theta_2-\xi^5\log\xi\right\}$$

$$\left.+\frac{C_4}{6}\Theta_2^6\left\{(1-\xi^6)\log\Theta_2-\xi^6\log\xi\right\}\right]+\frac{\eta_1^2}{2}\Theta_2^2\left[(\log\Theta_2)^2-\xi^2(\log\Theta_1)^2\right]$$

neglect the dimension of Θ_2 [which is equal for any time]

$$① = 0.32\left\{C_0^2+\frac{1}{2}\eta_1^2+\eta_1C_0\right\}+0.261333\left\{2C_0C_1+\frac{2}{3}C_1\eta_1\right\}$$

$$+\ 0.2176\left\{C_1^2+2C_0C_2+\frac{1}{2}C_2\eta_1\right\}+0.184448\left\{2C_0C_3+2C_1C_2+\frac{2}{5}C_3\eta_1\right\}$$

$$+\ 0.158891\left\{C_2^2+2C_0C_4+2C_1C_3+\frac{1}{3}C_4\eta_1\right\}+0.1388581\left\{2C_1C_4+2C_2C_3\right\}$$

$$+\ 0.12290048\left\{C_3^2+2C_2C_4\right\}+0.1099914\left\{2C_3C_4\right\}+0.09395338\left\{C_4^2\right\}$$

$$+2\eta_1\left[0.644878\left(C_0+\frac{\eta_1}{2}\right)+0.564960\,C_1+0.484492\,C_2+0.416763\,C_3\right.$$

$$\left.+\ 0.361887\,C_4\right]+\ 1.22620\,\eta_1^2$$

161)

$$\text{(I)} = \eta_1 \Big[2.03108\, \eta_1 + 1.609756\, C_0 + 1.304140\, C_1 + 1.077784\, C_2$$
$$+ 0.907305\, C_3 + 0.776738\, C_4 \Big]$$
$$+ C_0 \Big[0.32\, C_0 + 0.522667\, C_1 + 0.4352\, C_2 + 0.368896\, C_3 + 0.317782\, C_4 \Big]$$
$$+ C_1 \Big[0.2176\, C_1 + 0.368896\, C_2 + 0.317782\, C_3 + 0.2777162\, C_4 \Big]$$
$$+ C_2 \Big[0.158891\, C_2 + 0.2777162\, C_3 + 0.24580096\, C_4 \Big]$$
$$+ C_3 \Big[0.12290048\, C_3 + 0.2199828\, C_4 \Big] + 0.09395338\, C_4^2$$

$$\Theta_2 = 0.10$$

$$\text{(I)} = (1.10742\gamma^2 - 0.649688\gamma + 0.0952425)(20.97831\gamma^2 - 10.87918\gamma + 1.414631)$$
$$-(2.21856\gamma^2 - 1.88817\gamma + 0.205107)(8.51832\gamma^2 - 4.44821\gamma + 0.568532)$$
$$-(2.55125\gamma^2 - 1.36125\gamma + 0.18150)(8.62622\gamma^2 - 4.58885\gamma + 0.574549)$$
$$+(10.24073\gamma^2 - 5.82416\gamma + 0.799987)(6.27525\gamma^2 - 3.31278\gamma + 0.415217)$$
$$+(13.2422\gamma^2 - 6.73125\gamma + 0.852583)(2.42293\gamma^2 - 1.28318\gamma + 0.150721)$$
$$+(3.62549\gamma^2 - 2.07246\gamma + 0.208828)^2 \times 0.0939533\gamma$$

$$\text{I} = -(1.107\gamma^2 - 0.650\gamma + 0.0953)(4.539\gamma^2 - 2.089\gamma + 0.2243)$$
$$+(2.719\gamma^2 - 1.488\gamma + 0.2051)(1.857\gamma^2 - 0.823\gamma + 0.1017)$$
$$+(2.531\gamma^2 - 1.361\gamma + 0.1815)(0.3115\gamma^2 + 0.0470\gamma - 0.00761)$$
$$-(10.740\gamma^2 - 5.578\gamma + 0.7220)(1.535\gamma^2 - 0.606\gamma + 0.07624)$$
$$+(14.883\gamma^2 - 7.206\gamma + 0.8724)(1.031\gamma^2 - 0.430\gamma + 0.0613) + 0.09395\ C_4^2$$

$-5.025\gamma^4$	$+2.313\gamma^3$	$-0.2483\gamma^2$		
	$+2.950\gamma^3$	$-1.3579\gamma^2$	$+0.1458\gamma$	
		$-0.4326\gamma^2$	$+0.1991\gamma$	-0.02138
$+5.049\gamma^4$	$-2.238\gamma^3$	$+0.2765\gamma^2$		
	$-3.135\gamma^3$	$+1.3892\gamma^2$	-0.1716γ	
		$+0.3809\gamma^2$	-0.1688γ	$+0.02086$
$+0.788\gamma^4$	$+0.119\gamma^3$	$-0.0193\gamma^2$		
	$-0.424\gamma^3$	$-0.0640\gamma^2$	$+0.0104\gamma$	
		$+0.0565\gamma^2$	$+0.0085\gamma$	-0.00138
$-16.486\gamma^4$	$+6.508\gamma^3$	$-0.8188\gamma^2$		
	$+8.562\gamma^3$	$-3.3803\gamma^2$	$+0.4253\gamma$	
		$-1.1083\gamma^2$	$+0.4375\gamma$	-0.05505
$+15.344\gamma^4$	$-6.399\gamma^3$	$+0.9123\gamma^2$		
	$-7.429\gamma^3$	$+3.099\gamma^2$	-0.4417γ	
		$+0.899\gamma^2$	-0.3751γ	$+0.05348$
$1.235\gamma^4$	$-1.412\gamma^3$	$+0.5458\gamma^2$	-0.0813γ	$+0.00404$
$+0.905\gamma^4$	$-0.585\gamma^3$	$+0.1297\gamma^2$ $+0.0393$	-0.0119γ	$+0.00057$
-0.069	-0.28		-0.0108	$+0.0086$
0.836	-0.613	I 0.1690	-0.0227	$+0.00133$

163)

$$2.248\gamma^2 - 1.320\gamma + 0.1936$$
$$-4.378\gamma^2 + 2.718\gamma - 0.3302$$
$$-3.300\gamma^2 + 1.775\gamma - 0.2367$$
$$+11.578\gamma^2 - 6.013\gamma + 0.7783$$
$$-13.503\gamma^2 + 6.538\gamma - 0.7915$$
$$+2.816\gamma^2 - 1.609\gamma + 0.1622$$

$$-0.870\gamma^2 + 0.540\gamma - 0.0656$$
$$-1.323\gamma^2 + 0.711\gamma - 0.0949$$
$$+4.674\gamma^2 - 2.428\gamma + 0.3142$$
$$-5.490\gamma^2 + 2.658\gamma - 0.3218$$
$$+1.152\gamma^2 - 0.658\gamma + 0.0664$$

$$-0.5507\gamma^2 + 0.2962\gamma - 0.03949$$
$$+3.9620\gamma^2 - 2.0577\gamma + 0.26635$$
$$-4.7295\gamma^2 + 2.2899\gamma - 0.27723$$
$$+1.0067\gamma^2 - 0.5754\gamma + 0.05798$$

$$+1.7065\gamma^2 - 0.8863\gamma + 0.11472$$
$$-4.1330\gamma^2 + 2.0013\gamma - 0.24228$$
$$+0.8910\gamma^2 - 0.5093\gamma + 0.05132$$

$$-1.8291\gamma^2 + 0.8856\gamma - 0.10722$$
$$+0.7975\gamma^2 - 0.4557\gamma + 0.04594$$

$$C_5 = A - \frac{\eta}{2} - k_1\Theta_2^2 - k_2\Theta_2^3 + \frac{1}{2}(2k_1\Theta_2 + 3k_2\Theta_2^2)^2$$

$$= \Theta_2^2\Big[-(1873.44\gamma^2 - 1133.23\gamma + 142.774) - (553.71\gamma^2 - 324.844\gamma + 47.64375)$$
$$+ 12.500\gamma - 4.500 + (1582.03125\gamma^2 - 928.125\gamma + 136.125)\Big]$$

$$\boxed{\frac{C_5}{\Theta_2^2} = -(845.12\gamma^2 - 542.44\gamma + 58.313)}$$

$$C_6 = \Theta_2\Big[(16.50 - 56.25\gamma)(1 - 37.50\gamma + 6 - 45.00 + 187.5\gamma)$$
$$- \frac{2}{3}\eta_2\Big]$$

$$= \Theta_2\Big[(16.50 - 56.25\gamma)(150\gamma - 38) + \frac{2}{3}(5906.25\gamma^2 - 3195.00\gamma + 429)\Big]$$

$$= \Theta_2\Big[-(8440\gamma^2 - 4614\gamma + 627) + (3940\gamma^2 - 2132\gamma + 286)\Big]$$

$$\boxed{\frac{C_6}{\Theta_2} = -(4500\gamma^2 - 2482\gamma + 341.0)}$$

$$C_7 = \Big[-\frac{3}{8}\eta_3 + (75.00\gamma - 19.50)(150\gamma - 38) + (22.50 - 93.75\gamma)(16.50 - 56.25\gamma)\Big]$$

$$= (11250\gamma^2 - 5775\gamma + 741) + (5273\gamma^2 - 2813\gamma + 371.3)$$
$$-(4335\gamma^2 - 2200\gamma + 278)$$

$$\boxed{C_7 = (12188\gamma^2 - 6388\gamma + 834)}$$

$$\Theta_2 C_8 = -\frac{4}{15}\eta_4 - 7.500 + 31.25\gamma - (14062.5\gamma^2 - 6968.75\gamma + 862.5) \quad 165)$$

$$= -(14062.5\gamma^2 - 6968.75\gamma + 862.5) - 7.500 + 31.25\gamma - (656\gamma^2 - 190\gamma + 7.93)$$

$$\boxed{\Theta_2 C_8 = -(14719\gamma^2 - 7190\gamma + 878)}$$

$$\Theta_2^2 C_9 = \frac{1}{2}\left[(3k_2)^2 - \frac{5}{12}\eta_5\right]$$

$$= \frac{1}{2}\left[(506.25 - 4218.75\gamma + 8789.0625\gamma^2) - (1280\gamma^2 - 61.5\gamma + 73.9)\right]$$

$$\boxed{\Theta_2^2 C_9 = (3755\gamma^2 - 2079\gamma + 216.0)}$$

$$\frac{C_{10}}{\Theta_2^2} = -(1873\gamma^2 - 1133\gamma + 142.3) - (19\gamma - 3) - (7.5 - 31\gamma)$$

$$= -(1873\gamma^2 - 1133\gamma + 142.3) + 12\gamma - 4.5$$

$$\boxed{\frac{C_{10}}{\Theta_2^2} = -(1873\gamma^2 - 1145\gamma + 146.8)}$$

$$\frac{C_{11}}{\Theta_2} = (16.50 - 56.25\gamma) + (1968\gamma^2 - 1065\gamma + 143)$$

$$\boxed{\frac{C_{11}}{\Theta_2} = 1968\gamma^2 - 1121\gamma + 159.5}$$

$$C_{12} = (75.00\gamma - 19.5) - (1446\gamma^2 - 733.6\gamma + 92.69)$$

$$\boxed{C_{12} = -(1446\gamma^2 - 808.6\gamma + 112.19)}$$

$$\Theta_2 C_{13} = -\frac{\eta_4}{15} + k_2 = 7.500 - 31.25\gamma - (164.06\gamma^2 - 47.5\gamma + 1.95) \qquad (166)$$

$$\boxed{\Theta_2 C_{13} = -(164.06\gamma^2 - 16.25\gamma - 5.55)}$$

$$\boxed{\Theta_2^2 C_{14} = -(128.17\gamma^2 - 6.152\gamma + 7.38)}$$

$$
\begin{aligned}
(\mathrm{II}) = {} & \frac{1}{2}\left(1 - \frac{1}{\xi^2}\right)B^2 + B(C_5 - C_{10})(\log\Theta_2 - \log\Theta_1) + (1-\xi)B(C_6 - C_{11}) \\
& + \frac{1}{2}(1-\xi^2)\left\{BC_7 + C_5C_{10} - BC_{12} + \frac{\eta_1^2}{8} + \frac{\eta_1}{4}(C_5 + C_{10})\right\} \\
& + \frac{1}{3}(1-\xi^3)\left\{B(C_8 - C_{13}) + C_6C_{10} + C_5C_{11} + \frac{\eta_1}{6}(C_6 + C_{11})\right\} \\
& + \frac{1}{4}(1-\xi^4)\left\{B(C_9 - C_{14}) + C_7C_{10} + C_6C_{11} + C_5C_{12} + \frac{\eta_1}{8}(C_7 + C_{12})\right\} \\
& + \frac{1}{5}(1-\xi^5)\left\{C_8C_{10} + C_7C_{11} + C_6C_{12} + C_5C_{13} + \frac{\eta_1}{10}(C_8 + C_{13})\right\} \\
& + \frac{1}{6}(1-\xi^6)\left\{C_9C_{10} + C_8C_{11} + C_7C_{12} + C_6C_{13} + C_5C_{14} + \frac{\eta_1}{12}(C_9 + C_{14})\right\} \\
& + \frac{1}{7}(1-\xi^7)\left\{C_9C_{11} + C_8C_{12} + C_7C_{13} + C_6C_{14}\right\} \\
& + \frac{1}{8}(1-\xi^8)\left\{C_9C_{12} + C_8C_{13} + C_7C_{14}\right\} \\
& + \frac{1}{9}(1-\xi^9)\left\{C_9C_{13} + C_8C_{14}\right\} + \frac{1}{10}(1-\xi^{10})(C_9C_{14}) \\
& + \frac{\eta_1}{2}\Big\{\left(C_5 + C_{10} + \frac{\eta_1}{2}\right)0.644878 + (C_6 + C_{11})0.564960 + (C_7 + C_{12})0.484492 \\
& + (C_8 + C_{13})0.416763 + (C_9 + C_{14})0.361887\Big] + 0.306550\,\eta_1^2
\end{aligned}
$$

$$\text{II} = -0.888889B^2 + 0.510825B(C_5 - C_{10}) + \underline{\underline{0.6}}B(C_6 - C_{11}) \quad 167)$$

$$+0.32\left\{B(C_7 - C_{12}) + C_5C_{10} + \frac{\eta_1^2}{8} + \frac{\eta_1}{4}(C_5 + C_{10})\right\}$$

$$+0.261333\left\{B(C_8 - C_{13}) + C_6C_{10} + C_5C_{11} + \frac{\eta_1}{6}(C_6 + C_{11})\right\}$$

$$+0.2176\left\{B(C_9 - C_{14}) + C_7C_{10} + C_6C_{11} + C_5C_{12} + \frac{\eta_1}{8}(C_7 + C_{12})\right\}$$

$$+0.184448\left\{C_8C_{10} + C_7C_{11} + C_6C_{12} + C_5C_{13} + \frac{\eta_1}{10}(C_8 + C_{13})\right\}$$

$$+0.158891\left\{C_9C_{10} + C_8C_{11} + C_7C_{12} + C_6C_{13} + C_5C_{14} + \frac{\eta_1}{12}(C_9 + C_{14})\right\}$$

$$+0.1388581\left\{C_9C_{11} + C_8C_{12} + C_7C_{13} + C_6C_{14}\right\}$$

$$+0.12290048\left\{C_9C_{12} + C_8C_{13} + C_7C_{14}\right\}$$

$$+0.1099914\left\{C_9C_{13} + C_8C_{14}\right\} + 0.09395338\,C_9C_{14}$$

$$+\frac{\eta_1}{2}\left\{0.644878\left(C_5 + C_{10} + \frac{\eta_1}{2}\right) + 0.564960(C_6 + C_{11}) + 0.484492(C_7 + C_{12})\right.$$

$$\left. +0.416763(C_8 + C_{13}) + 0.361887(C_9 + C_{14})\right\} + 0.306550\,\eta_1^2$$

$$= B\left[-0.888889B + 0.510825(C_5 - C_{10}) + 0.6(C_6 - C_{11}) + 0.32(C_7 - C_{12})\right.$$

$$\left. + 0.261333(C_8 - C_{13}) + 0.2176(C_9 - C_{14})\right]$$

$$+\eta_1\left[\cancel{0.04\eta_1 + 0.08(C_5 + C_{10}) + 0.0435555(C_6 + C_{11}) + 0.0222(C_7 + C_{12})}\right.$$

$$\left.\cancel{+0.0184448(C_8 + C_{13}) + 0.0132409(C_9 + C_{14})}\right]$$

$$+ C_5\left[0.32C_{10} + 0.261333C_{11} + 0.2176C_{12} + 0.184448C_{13} + 0.158891C_{14}\right]$$

$$+ C_6\left[0.261333C_{10} + 0.2176C_{11} + 0.184448C_{12} + 0.158891C_{13} + 0.1388581C_{14}\right]$$

$$+ C_7\left[0.2176C_{10} + 0.184448C_{11} + 0.158891C_{12} + 0.1388581C_{13} + 0.12290048C_{14}\right]$$

$$
\begin{aligned}
&+ C_8\left[0.184448\, C_{10} + 0.158891\, C_{11} + 0.1388581\, C_{12} + 0.12290048\, C_{13} + 0.1099914\, C_{14}\right] \\
&+ C_9\left[0.158891\, C_{10} + 0.1388581\, C_{11} + 0.12290048\, C_{12} + 0.1099914\, C_{13} + 0.09395338\, C_{14}\right] \\
&+ \eta_1\left\{0.507770\, \eta_1 + 0.402439(C_5 + C_{10}) + 0.326036(C_6 + C_{11}) + 0.269446(C_7 + C_{12}) \right. \\
&\left. + 0.226827(C_8 + C_{13}) + 0.194185(C_9 + C_{14})\right\}
\end{aligned} \quad 1.68)
$$

$$\boxed{C_5 - C_{10} = +(1028\gamma^2 - 603\gamma + 88.5)}$$

$$\boxed{C_5 + C_{10} = -(2718\gamma^2 - 1687\gamma + 205.1)}$$

$$\boxed{C_6 - C_{11} = -(6468\gamma^2 - 3603\gamma + 500.5)}$$

$$\boxed{C_6 + C_{11} = -(2532\gamma^2 - 1361\gamma + 181.5)}$$

$$\boxed{C_7 - C_{12} = +(13636\gamma^2 - 7197\gamma + 946)}$$

$$\boxed{C_7 + C_{12} = +(10742\gamma^2 - 5579\gamma + 722)}$$

$$\boxed{C_8 - C_{13} = -(14555\gamma^2 - 7174\gamma + 872)}$$

$$\boxed{C_8 + C_{13} = -(14883\gamma^2 - 7206\gamma + 884)}$$

$$\boxed{C_9 - C_{14} = +(3883\gamma^2 - 2085\gamma + 223.4)}$$

$$\boxed{C_9 + C_{14} = +(3627\gamma^2 - 2073\gamma + 208.6)}$$

169)

$$
\begin{aligned}
\mathrm{II} = & -(0.1146\gamma^2 - 0.04981\gamma + 0.005831)(1.1471\gamma^2 - 1.0162\gamma + 0.13682) \\
& -(1.1074\gamma^2 - 0.6497\gamma + 0.09529)(1.0493\gamma^2 - 0.4729\gamma + 0.05392) \\
& +(0.8451\gamma^2 - 0.5424\gamma + 0.05831)(0.4503\gamma^2 - 0.2534\gamma + 0.02993) \\
& +(4.500\gamma^2 - 2.482\gamma + 0.3410)(0.3718\gamma^2 - 0.2078\gamma + 0.02625) \\
& -(12.188\gamma^2 - 6.388\gamma + 0.834)(0.3104\gamma^2 - 0.1739\gamma + 0.02303) \\
& +(14.719\gamma^2 - 7.190\gamma + 0.878)(0.2679\gamma^2 - 0.1482\gamma + 0.01881) \\
& -(3.755\gamma^2 - 2.079\gamma + 0.2160)(0.2320\gamma^2 - 0.1280\gamma + 0.01627) \\
= & \boxed{0.0493\gamma^4 + 0.0203\gamma^3 - 0.0281\gamma^2 + 0.00770\gamma - 0.000612}
\end{aligned}
$$

170)

$$-0.1019z^2+0.04428z-0.005183$$
$$+0.5251z^2-0.3080z+0.04521$$
$$-3.8808z^2+2.1618z-0.30030$$
$$+7.5635z^2-2.3030z+0.30272$$
$$-3.80370z^2+1.8746z-0.22788$$
$$+0.8449z^2-0.45370z+0.04861$$

$$0.56230z^2-0.3299z+0.04839$$
$$-1.09383z^2+0.6789z-0.08254$$
$$-0.82552z^2+0.4437z-0.05918$$
$$+2.8944z^2-1.5032z+0.19454$$
$$-3.3759z^2+1.6345z-0.20052$$
$$+0.7892z^2-0.4511z+0.04539$$

$$-0.5994z^2+0.3664z-0.04698$$
$$+0.5143z^2-0.2930z+0.04168$$
$$-0.3146z^2+0.1780z-0.02441$$
$$-0.0303z^2+0.0030z-0.0010$$
$$-0.0203z^2+0.0010z-0.0012$$

$$-0.4895z^2+0.2992z-0.03836$$
$$+0.4282z^2-0.2439z+0.03471$$
$$-0.2667z^2+0.1491z-0.02070$$
$$-0.0261z^2+0.0026z-0.00088$$
$$-0.0177z^2+0.0008z-0.00102$$

$$-0.4076z^2+0.2492z-0.03194$$
$$+0.3630z^2-0.2068z+0.02942$$
$$-0.2298z^2+0.1285z-0.01783$$
$$-0.0203z^2+0.0023z-0.00077$$
$$-0.0157z^2+0.0007z-0.00091$$

$$-0.3455z^2+0.2112z-0.02708$$
$$+0.3127z^2-0.1781z+0.02534$$
$$-0.2008z^2+0.1123z-0.01558$$
$$-0.0202z^2+0.0020z-0.00068$$
$$-0.0141z^2+0.0008z-0.00081$$

$$-0.2976z^2+0.1819z-0.02333$$
$$+0.2733z^2-0.1557z+0.02215$$
$$-0.1777z^2+0.0994z-0.01379$$
$$-0.0180z^2+0.0018z-0.00061$$
$$-0.0120z^2+0.0006z-0.00069$$

$-0.13145z^4$	$+0.11646z^3$	$-0.015680z^2$	—	—
—	$+0.05714z^3$	$-0.050617z^2$	$+0.006815z$	—
—	—	$-0.006689z^2$	$+0.005925z$	-0.0007978
$-1.1620z^4$	$+0.52369z^3$	$-0.059711z^2$	—	—
—	$+0.68173z^3$	$-0.30724z^2$	$+0.035032z$	—
—	—	$-0.09999z^2$	$+0.04506z$	-0.005138
$+0.3805z^4$	$-0.21415z^3$	$+0.02529z^2$	—	—
—	$-0.24424z^3$	$+0.13744z^2$	$-0.016234z$	—
—	—	$+0.02625z^2$	$-0.014778z$	$+0.001745$
$+1.6735z^4$	$-0.93510z^3$	$+0.118125z^2$	—	—
—	$-0.92281z^3$	$+0.515760z^2$	$-0.065153z$	—
—	—	$+0.12678z^2$	$-0.070860z$	$+0.008951$
$-3.7832z^4$	$+2.11949z^3$	$-0.26850z^2$	—	—
—	$+1.98284z^3$	$-1.11087z^2$	$+0.14073z$	—
—	—	$-0.2589z^2$	$+0.14503z$	-0.018373
$+3.9432z^4$	$-2.18136z^3$	$+0.27686z^2$	—	—
—	$-1.9262z^3$	$+1.0656z^2$	$-0.13524z$	—
—	—	$+0.2352z^2$	$-0.13012z$	$+0.016515$
$-0.8712z^4$	$+0.48064z^3$	$-0.06109z^2$	—	—
—	$+0.48233z^3$	$-0.26611z^2$	$+0.033825z$	—
—	—	$-0.05011z^2$	$+0.027648z$	-0.003514

$$\boxed{\frac{G_1}{\Theta_2} = (56.25\gamma - 16.50)}$$

$$G_2 = 2A + 4(k_1 + 3k_2\Theta_2) - \frac{\gamma}{2}$$

$$= -(37.4688\gamma^2 - 22.6646\gamma + 2.84588) - (5.5371\gamma^2 - 3.248447\gamma + 0.4764375)$$

$$+ 78.00 - 300\gamma$$

$$\boxed{G_2 = -(43.0059\gamma^2 + 274.087\gamma - 74.6777)}$$

$$\Theta_2 G_3 = -67.5 + 281.25\gamma + 59.0625\gamma^2 - 31.95\gamma + 4.290$$

$$\boxed{\Theta_2 G_3 = 59.0625\gamma^2 + 249.3\gamma - 63.21}$$

$$G_4 = 39.00 - 150.00\gamma - (18.7344\gamma^2 - 11.3323\gamma + 1.42294)$$

$$-(5.5371\gamma^2 - 3.24844\gamma + 0.47644)$$

$$\boxed{G_4 = -(24.2715\gamma^2 + 135.419\gamma - 37.1007)}$$

$$\Theta_2 G_5 = 45.00 - 187.5\gamma - (39.3750\gamma^2 - 21.300 + 2.86)$$

$$\boxed{\Theta_2 G_5 = -(39.3750\gamma^2 + 166.2\gamma - 42.14)}$$

$$\boxed{\frac{G_6}{\Theta_2} = 16.50 - 56.25\gamma}$$

$$\boxed{G_7 = -(18.7344\gamma^2 + 138.668\gamma - 37.5771)}$$

$$G_8 = \frac{1}{2} G_5$$

$$\frac{1}{\Theta_2}\,\text{(III)} = G_1^2 \log\xi + (1-\xi)[2G_1G_2 + 2\eta_1 G_1] + \tfrac{1}{2}(1-\xi^2)[G_2^2 + 2G_1G_3 + \eta_1 G_2] \qquad 173)$$

$$+\tfrac{1}{3}(1-\xi^3)[2G_2G_3 - \eta_3 G_1 + \tfrac{2}{3}\eta_1 G_3] + \tfrac{1}{4}(1-\xi^4)\left[G_3^2 - \eta_4 G_1 - \eta_3 G_2 - \frac{\eta_1\eta_3}{4}\right]$$

$$-\tfrac{1}{5}(1-\xi^5)[\eta_4 G_2 + \eta_3 G_3 + \eta_5 G_1 + \tfrac{2}{15}\eta_1\eta_4]$$

$$-\tfrac{1}{6}(1-\xi^6)\left[\eta_4 G_3 + \eta_5 G_2 - \frac{\eta_3^2}{4} + \frac{1}{12}\eta_1\eta_5\right] - \tfrac{1}{7}(1-\xi^7)\left[\eta_5 G_3 - \frac{\eta_3\eta_4}{3}\right]$$

$$+\tfrac{1}{8}(1-\xi^8)\left[\frac{\eta_4^2}{9} + \frac{\eta_3\eta_5}{4}\right] + \tfrac{1}{9}(1-\xi^9)\left[\frac{\eta_4\eta_5}{6}\right] + \tfrac{1}{10}(1-\xi^{10})\frac{\eta_5^2}{16}$$

$$-2\eta_1\left[G_1\{(1-\xi)\log\Theta_2 - \xi\log\xi\} + \text{- - - - - - -}\right]$$

$$\text{(III)} = 0.510825\,G_1^2 + 0.4[2G_1G_2 + 2\eta_1 G_1] + 0.32[G_2^2 + 2G_1G_3 + \eta_1 G_2]$$

$$+ 0.261333[2G_2G_3 - \eta_3 G_1 + \tfrac{2}{3}\eta_1 G_3] + 0.2176\left[G_3^2 - \eta_4 G_1 - \eta_3 G_2 - \frac{\eta_1\eta_3}{4}\right]$$

$$- 0.184448[\eta_4 G_2 + \eta_3 G_3 + \eta_5 G_1 + \tfrac{2}{15}\eta_1\eta_4] - 0.158891\left[\eta_4 G_3 + \eta_5 G_2 - \frac{\eta_3^2}{4} + \frac{1}{12}\eta_1\eta_5\right]$$

$$- 0.1388581\left\{\eta_5 G_3 - \frac{\eta_3\eta_4}{3}\right\} + 0.12290048\left[\frac{\eta_4^2}{9} + \frac{\eta_3\eta_5}{4}\right]$$

$$+ 0.1099914\left[\frac{\eta_4\eta_5}{6}\right] + 0.0939534\left[\frac{\eta_5^2}{16}\right]$$

$$+2\eta_1\left[0.614539\,G_1 + 0.644878\,G_2 + 0.564960\,G_3 - 0.484492\,\frac{\eta_3}{2}\right.$$

$$\left. - 0.416763\,\frac{\eta_4}{3} - 0.361887\,\frac{\eta_5}{4}\right]$$

$$= \eta_1[2.029078\,G_1 + 1.609756\,G_2 + 1.304142\,G_3 - 0.538892\,\eta_3$$

$$- 0.302435\,\eta_4 - 0.194185\,\eta_5]$$

$$+ G_1[0.510825\,G_1 + 0.8\,G_2 + 0.64\,G_3 - 0.261333\,\eta_3 - 0.2176\,\eta_4$$

$$- 0.184448\,\eta_5]$$

$$+ G_2[0.32\,G_2 + 0.522667\,G_3 - 0.2176\,\eta_3 - 0.184448\,\eta_4 - 0.158891\,\eta_5]$$

174)

$$+G_3\left[0.2176\,G_3-0.184448\,\eta_3-0.158891\,\eta_4-0.1388581\,\eta_5\right]$$

$$+\eta_3\left[0.0397228\,\eta_3+0.0462860\,\eta_4+0.030725\,\eta_5\right]$$

$$+\eta_4\left[0.0136556\,\eta_4+0.0183319\,\eta_5\right]+0.0058721\,\eta_5^2$$

$$\text{III}=-(1.1074\gamma^2-0.64969\gamma+0.095288)(6.7949\gamma^2-3.2112\gamma+0.0130)$$

$$-(5.625\gamma-1.650)(3.7861\gamma^2+1.3825\gamma-0.8530)$$

$$+(4.30059\gamma^2+27.4087\gamma-7.46777)(1.7488\gamma^2-5.6912\gamma+1.1090)$$

$$-(5.90625\gamma^2+24.93\gamma-6.321)(1.6664\gamma^2-6.6410\gamma+1.5415\gamma)$$

$$+(1.15664\gamma^2-0.586875\gamma+0.07415)(6.6788\gamma^2-2.7064\gamma+0.3624)$$

$$+(0.24609\gamma^2-0.07125\gamma+0.002925)(0.9000\gamma^2-0.1244\gamma+0.03647)$$

$$+0.0058721\,\eta_5^2$$

$$=-1.8444\gamma^4+0.6608\gamma^3+0.9622\gamma^2-0.5336\gamma+0.0805$$

(174 a)

$$\begin{array}{l|l}
+11.4136\gamma - 3.34798 & +2.87339\gamma - 0.84286 \\
-6.92292\gamma^2 - 44.1214\gamma + 12.0213 & -3.44047\gamma^2 - 21.92696\gamma + 5.97422 \\
+7.70258\gamma^2 + 32.5122\gamma - 8.2435 & +3.78000\gamma^2 + 15.95520\gamma - 4.04544 \\
-6.23304\gamma^2 + 3.1626\gamma - 0.3996 & -3.02268\gamma^2 + 1.53367\gamma - 0.19378 \\
-0.74426\gamma^2 + 0.2155\gamma - 0.0088 & -0.53549\gamma^2 + 0.15504\gamma - 0.00636 \\
-0.59731\gamma^2 + 0.2867\gamma - 0.0344 & -0.56736\gamma^2 + 0.27235\gamma - 0.03268 \\
\hline
-1.37619\gamma^2 - 8.77078\gamma + 2.38969 & 1.28520\gamma^2 + 5.42477\gamma - 1.37545 \\
+3.08700\gamma^2 + 13.03009\gamma - 3.30378 & -2.13340\gamma^2 + 1.08248\gamma - 0.13677 \\
-2.51685\gamma^2 + 1.27704\gamma - 0.16136 & -0.391015\gamma^2 + 0.11321\gamma - 0.00464 \\
-0.45390\gamma^2 + 0.13142\gamma - 0.0053 & -0.42715\gamma^2 + 0.20504\gamma - 0.02461 \\
-0.48878\gamma^2 + 0.23462\gamma - 0.0282 & \\
\hline
4.5945\gamma^2 - 2.33123\gamma + 0.2945 & 0.33605\gamma^2 - 0.09730\gamma + 0.003994 \\
1.1391\gamma^2 - 0.3298\gamma + 0.0135 & 0.56393\gamma^2 - 0.27707\gamma + 0.03248 \\
0.94516\gamma^2 - 0.4537\gamma + 0.0544 &
\end{array}$$

28

$$\eta_5^2 \times 10^{-4} = \frac{0.09462\gamma^4 - 0.009084\gamma^3 + \overset{0.0109006}{0.0002160}\,\gamma - 0.000523}{0.09462\gamma^4 - 0.009084\gamma^3 + 0.0111186\gamma - 0.000523\gamma}$$

$+ 0.$

$-7.5247\gamma^4$	$+3.5561\gamma^3$	$-0.0144\gamma^2$		
	$+4.4146\gamma^3$	$-2.0863\gamma^2$	$+0.00845\gamma$	
		$-0.6475\gamma^2$	$+0.30599\gamma$	-0.001239
	$-21.2968\gamma^3$	$-7.7766\gamma^2$	$+4.79813\gamma$	
		$+6.2470\gamma^2$	$+2.28113\gamma$	-1.40745
$+7.52087\gamma^4$	$-24.4755\gamma^3$	$+4.7694\gamma^2$		
	$+47.9323\gamma^3$	$-155.9884\gamma^2$	$+30.3962\gamma$	
		$-13.0596\gamma^2$	$+42.5006\gamma$	-8.28176
$-9.84218\gamma^4$	$+39.2234\gamma^3$	$-9.10448\gamma^2$		
	$-41.5434\gamma^3$	$+165.5601\gamma^2$	-38.4296γ	
		$+10.5333\gamma^2$	-41.9778γ	$+9.74382$
$+7.72450\gamma^4$	$-3.13033\gamma^3$	$+0.41917\gamma^2$		
	$-3.91962\gamma^3$	$+1.58831\gamma^2$	-0.21288γ	
		$+0.49523\gamma^2$	-0.20068γ	$+0.02887$
$+0.221481\gamma^4$	$-0.030614\gamma^3$	$+0.00897\gamma^2$		
	$-0.06413\gamma^3$	$+0.00886\gamma^2$	-0.002598γ	
		$+0.00263\gamma^2$	-0.000364γ	$+0.000107$
$+0.05556\gamma^4$	$-0.105334\gamma^3$	$+0.006528\gamma^2$	-0.000307γ	$+0.000184$

$$\boxed{IV} = \frac{1}{2}\left(1-\frac{1}{\xi^2}\right)B^2 - \left(1-\frac{1}{\xi}\right)BG_6 + B(G_4-G_7)\log\frac{1}{\xi}$$ 176)

$$= -0.88888889\,B^2 - 0.666667\,BG_6 + 0.510825\,B(G_4-G_7) \quad \left(+\frac{1}{8}\eta_1^2\right)$$

$$-0.4\left\{B(G_5-G_8)+G_4G_6+\frac{\eta_1}{2}G_6\right\}+0.32\left\{G_4G_7+G_5G_6-\frac{\eta_3 B}{2}+\frac{\eta_1}{4}(G_4+G_7)\right\}$$

$$+0.261333\left\{\frac{3\eta_3}{8}G_6-G_5G_7-G_8G_4-\frac{\eta_4}{5}B-\frac{\eta_1}{6}(G_5+G_8)\right\}$$

$$+0.2176\left\{G_5G_8-\frac{\eta_5}{6}B-\frac{4\eta_4}{15}G_6-\frac{\eta_3}{8}(3G_7+G_4)-\frac{\eta_1\eta_3}{16}\right\}$$

$$+0.184448\left\{\frac{\eta_3}{8}(3G_8+G_5)-\frac{5\eta_5}{24}G_6-\frac{\eta_4}{15}(4G_7+G_4)-\frac{\eta_1\eta_4}{30}\right\}$$

$$+0.158891\left\{\frac{\eta_4}{15}(4G_8+G_5)+\frac{3\eta_3^2}{64}-\frac{\eta_5}{24}(5G_7+G_4)-\frac{\eta_1\eta_5}{48}\right\}$$

$$+0.1388581\left\{\frac{\eta_5}{24}(5G_8+G_5)+\frac{7\eta_3\eta_4}{120}\right\}+0.1229005\left\{\frac{\eta_3\eta_5}{24}+\frac{4\eta_4^2}{225}\right\}$$

$$+0.1099914\left\{\frac{\eta_4\eta_5}{40}\right\}+0.09395338\left\{\frac{5}{576}\eta_5^2\right\}$$

$$-0.614539\,\frac{\eta_1G_6}{2}+0.644878\left\{\frac{\eta_1}{2}(G_4+G_7)+\left(\frac{\eta_1}{2}\right)^2\right\}-0.564960\,\frac{\eta_1}{2}(G_5+G_8)$$

$$-0.484492\,\frac{\eta_1\eta_3}{4}-0.416713\,\frac{\eta_1\eta_4}{6}-0.361887\,\frac{\eta_1\eta_5}{8}+0.30655\,\eta_1^2$$

$$= B\{-0.888889\,B-0.666667\,G_6+0.510825(G_4-G_7)-0.2\,G_5 \quad -0.16\,\eta_3$$

$$-0.0522666\,\eta_4-0.0362667\,\eta_5\}$$

$$+\ \eta_1\{-0.507269\,G_6+0.50777\,\eta_1+0.402439(G_4+G_7)-0.489053\,G_5 \quad -0.134723\,\eta_3$$

$$-0.0756088\,\eta_4-0.0485461\,\eta_5\}$$

177)

$$+ G_4 \left\{ -0.4\, G_6 + 0.32\, G_7 - 0.130667\, G_5 - 0.0272\, \eta_3 - 0.0122965\, \eta_4 \right.$$
$$\left. - 0.00662046\, \eta_5 \right\}$$

$$+ G_5 \left\{ 0.32\, G_6 - 0.261333\, G_7 + 0.1088\, G_5 + 0.023056\, \eta_3 + 0.0105927\, \eta_4 \right.$$
$$\left. + 0.00578575\, \eta_5 \right\}$$

$$+ \eta_3 \left\{ 0.0979999\, G_6 - 0.0816\, G_7 + 0.034584\, G_5 + 0.0074480\, \eta_3 + 0.0081001\, \eta_4 \right.$$
$$\left. + 0.0051209\, \eta_5 \right\}$$

$$+ \eta_4 \left\{ -0.0580267\, G_6 - 0.0491862\, G_7 + 0.0211855\, G_5 + 0.00218490\, \eta_4 \right.$$
$$\left. + 0.00274979\, \eta_5 \right\}$$

$$+ \eta_5 \left\{ -0.0384267\, G_6 - 0.0331023\, G_7 + 0.0144644\, G_5 + 0.00081557\, \eta_5 \right\}$$

$$\boxed{\begin{aligned} G_4 - G_7 &= -(5.5371\gamma^2 - 3.249\gamma + 0.4764) \\ G_4 + G_7 &= -(43.0059\gamma^2 + 274.087 - 74.6778) \end{aligned}}$$

$$G_5 - G_8 = \frac{1}{2} G_5$$

$$G_5 + G_8 = \frac{3}{2} G_5$$

$$\text{(IV)} - + (0.11459\gamma^2 - 0.049806\gamma + 0.0058314)(1.6880\gamma^2 - 8.1773\gamma + 2.0884) \quad 178)$$

$$-(1.10742\gamma^2 - 0.649688\gamma + 0.0952875)(1.1362\gamma^2 - 0.4731\gamma - 0.0451)$$

$$+(2.42715\gamma^2 + 13.5419\gamma - 3.71007)(0.45024\gamma^2 - 0.1537\gamma + 0.0299)$$

$$-(3.93250\gamma^2 + 16.62\gamma - 4.214)(0.37173\gamma^2 - 0.1281\gamma + 0.02291)$$

$$+(11.5664\gamma^2 - 5.86875\gamma + 0.741525)(0.13852\gamma^2 - 0.04407\gamma + 0.00748)$$

$$+(2.46094\gamma^2 - 0.71250\gamma + 0.02925)(0.022565\gamma^2 + 0.65439\gamma - 0.19075)$$

$$+(3.07617\gamma^2 - 0.14766\gamma + 0.177188)(0.007570\gamma^2 + 0.43466\gamma - 0.12670)$$

$$\boxed{= 0.2452\gamma^4 + 1.8989\gamma^3 - 0.7232\gamma^2 + 0.09285\gamma - 0.02039}$$

$+0.10186\gamma^2 - 0.044272\gamma + 0.0051835$
$+3.75000\gamma - 1.10000$
$-0.282848\gamma^2 + 0.16597\gamma - 0.024336$
$+0.7875\gamma^2 + 3.3240\gamma - 0.8428$
$-1.8506\gamma^2 + 0.9390\gamma - 0.11864$
$-0.12862\gamma^2 + 0.03724\gamma - 0.00153$
$-0.11156\gamma^2 + 0.00535\gamma - 0.00643$

$+0.562315\gamma^2 - 0.329892\gamma + 0.048384$ 179)
$+2.85339\gamma - 0.836994$
$-1.73051\gamma^2 - 11.03033\gamma + 3.00533$
$+1.92565\gamma^2 + 8.12806\gamma - 2.06087$
$-1.55826\gamma^2 + 0.79066\gamma - 0.09990$
$-0.18607\gamma^2 + 0.05387\gamma - 0.00221$
$-0.14933\gamma^2 + 0.00717\gamma - 0.00860$

$- \qquad +2.250\gamma - 0.6600$
$-0.59950\gamma^2 - 4.4374\gamma + 1.20247$
$+0.514501\gamma^2 + 2.17169\gamma - 0.55063$
$-0.314606\gamma^2 + 0.15963\gamma - 0.02017$
$-0.030211\gamma^2 + 0.00876\gamma - 0.00036$
$-0.020365\gamma^2 + 0.00098\gamma - 0.00117$

$-1.800\gamma + 0.528$
$+0.489592\gamma^2 + 3.62385\gamma - 0.982014$
$-0.42840\gamma^2 - 1.80823\gamma + 0.45848$
$+0.26667\gamma^2 - 0.13531\gamma + 0.01710$
$+0.026068\gamma^2 - 0.00755\gamma + 0.00031$
$+0.017798\gamma^2 - 0.00085\gamma + 0.00103$

$- \qquad -0.551249\gamma + 0.161700$
$+0.152873\gamma^2 + 1.13153\gamma - 0.30663$
$-0.136175\gamma^2 - 0.57479\gamma + 0.14574$
$+0.086147\gamma^2 - 0.043710\gamma + 0.005523$
$+0.019934\gamma^2 - 0.00577\gamma + 0.00024$
$+0.015753\gamma^2 - 0.00076\gamma + 0.00091$

$+0.32640\gamma - 0.0957441$
$+0.092147\gamma^2 + 0.682055\gamma - 0.184827$
$\pm 0.0834179\gamma^2 - 0.352103\gamma + 0.089276$
$+0.005377\gamma^2 - 0.001557\gamma + 0.00006$
$+0.008459\gamma^2 - 0.00041\gamma + 0.00048$

$+0.21615\gamma - 0.063404$
$+0.0620152\gamma^2 + 0.45902\gamma - 0.124319$
$-0.0569536\gamma^2 - 0.24039\gamma + 0.060953$
$+0.0025088\gamma^2 - 0.00012\gamma + 0.00014$

$+0.19343\gamma^4$	$-0.93704\gamma^3$	$+0.23931\gamma^2$	—	—
—	$-0.08407\gamma^3$	$+0.40729\gamma^2$	-0.10401γ	—
		$+0.00984\gamma^2$	-0.04769γ	$+0.012178$
$-1.2583\gamma^4$	$+0.52392\gamma^3$	$+0.04994\gamma^2$		
—	$+0.73818\gamma^3$	$-0.30737\gamma^2$	-0.02930γ	—
		$-0.10827\gamma^2$	$+0.04508\gamma$	$+0.004297$
$+1.0928\gamma^4$	$-0.37305\gamma^3$	$+0.07257\gamma^2$		
—	$+6.0971\gamma^3$	$-2.0814\gamma^2$	$+0.40490\gamma$	—
		$-1.6704\gamma^2$	$+0.57024\gamma$	-0.11093
$-1.4637\gamma^4$	$+0.5044\gamma^3$	$-0.0902\gamma^2$		
—	$-6.1782\gamma^3$	$+2.1290\gamma^2$	-0.38076γ	—
		$+1.5665\gamma^2$	-0.53981γ	$+0.09654$
$+1.6022\gamma^2$	$-0.5097\gamma^3$	$+0.0865\gamma^2$		
—	$-0.81294\gamma^3$	$+0.2576\gamma^2$	-0.04390γ	—
		$+0.1027\gamma^2$	-0.03268γ	$+0.00555$
$+0.0555\gamma^2$	$+1.6104\gamma^3$	$-0.4694\gamma^2$		
—	$-0.0161\gamma^3$	$-0.4663\gamma^2$	$+0.13591\gamma$	—
		$+0.0007\gamma^2$	$+0.01914\gamma$	-0.005579
$+0.0233\gamma^2$	$+1.3371\gamma^3$	$-0.3898\gamma^2$		
—	$-0.0011\gamma^3$	$-0.0642\gamma^2$	$+0.01871\gamma$	—
		$+0.0013\gamma^2$	$+0.07702\gamma$	-0.022449

Taking into account only the w for potential energy 181)

The integral gives

$$\int_0^{\Theta} 3w\cdot\theta\, d\theta$$

$$\cong \Theta_2^3\left\{\frac{\Theta_2\xi^2\gamma}{2}\right\} + \underbrace{\int_{\Theta_1}^{\Theta_2} 3w\theta\, d\theta}$$

$$3\int_{\Theta_1}^{\Theta_2} w\theta\, d\theta = 3\int_{\Theta_1}^{\Theta_2}\left[k_1(\Theta_2-\theta)^2 + k_2(\Theta_2-\theta)^3\right]\theta\, d\theta$$

$$= 3\int_{\Theta_1}^{\Theta_2}\left\{k_1\Theta_2^2 + k_2\Theta_2^3 - (2k_1\Theta_2 + 3k_2\Theta_2^2)\theta + (k_1 + 3k_2\Theta_2)\theta^2 - k_2\theta^3\right\}\theta\, d\theta.$$

$$= 3\left[\frac{1}{2}\theta^2\left\{k_1\Theta_2^2 + k_2\Theta_2^3\right\} - \frac{\theta^3}{3}\left\{2k_1\Theta_2 + 3k_2\Theta_2^2\right\} + \frac{\theta^4}{4}\left\{k_1 + 3k_2\Theta_2\right\} - \frac{\theta^5}{5}\left\{k_2\right\}\right]$$

$$= 3\Theta_2^3\left[0.32\left\{k_1\Theta_2 + k_2\Theta_2^2\right\} - 0.261333\left\{2k_1\Theta_2 + 3k_2\Theta_2^2\right\}\right.$$

$$\left. + 0.2176\left\{k_1\Theta_2 + 3k_2\Theta_2^2\right\} - 0.184448\left\{k_2\Theta_2^2\right\}\right]$$

$$= 3\Theta_2^3\left[0.032(4.5 - 12.5\gamma) - 0.0261333\left\{16.5 - 56.25\gamma\right\}\right.$$

$$\left. + 0.02176(19.5 - 75.00\gamma) - 0.0184448(7.50 - 31.25\gamma)\right]$$

$$= \Theta_2^3\left[0.0432\gamma - 0.00366\right]$$

182)

Total $\quad Q_2^3\left(0.0612\gamma - 0.00366\right)$

$$\therefore \quad \phi = \frac{3.344\gamma^3 - 1.839\gamma^2 + 0.3380\gamma - 0.0227}{0.0612}$$

$$\sigma = \frac{pr}{gt} = p \cdot 500 , \qquad \phi = \frac{p(1-\mu^2)}{3E} , \qquad p = \frac{3E}{(1-\mu^2)}\phi$$

$$\therefore \quad \sigma = 500 \quad \frac{3E}{(1-\mu^2)}\phi$$

$$\sigma_{classical} = \frac{E \times 0.001}{\sqrt{3(1-\mu^2)}}$$

$$\therefore \quad \frac{\sigma}{\sigma_{class}} = 500{,}000 \frac{3}{(1-\mu^2)} \sqrt{3(1-\mu^2)} \ \phi$$

$$= \frac{3\sqrt{3} \times 500{,}000\,\phi}{\sqrt{1-\mu^2}}$$

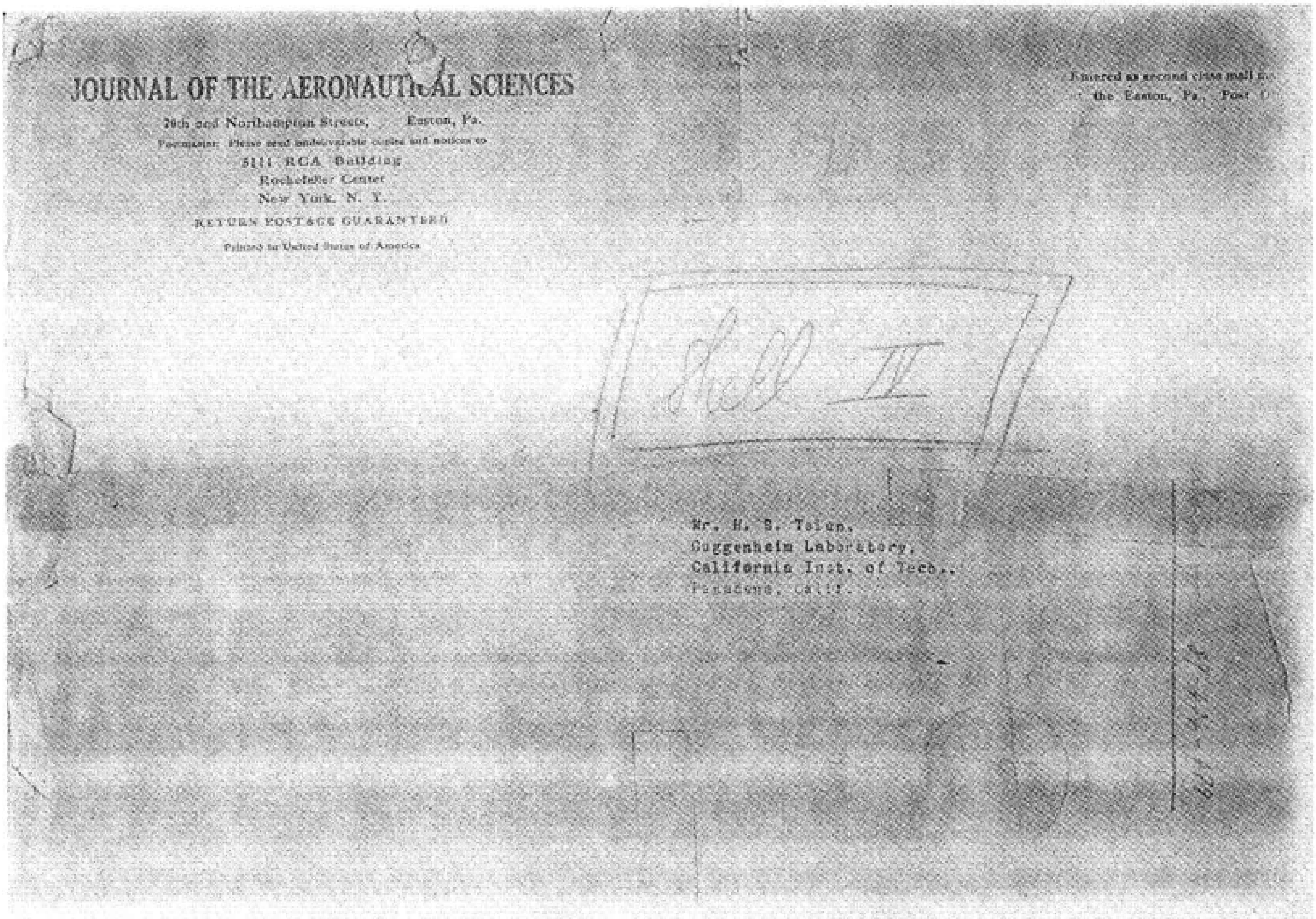
$$10.032\gamma^2 - 3.678\gamma + 0.3380 = 0$$

$$\gamma^2 - 0.3667\gamma + 0.03370 = 0$$

$$\gamma = \tfrac{1}{2}\left[0.3667 \pm \sqrt{0.3667^2 - 0.1348}\right] = \quad 0.1833$$

Section 3

Shell (Ⅲ) Preliminary Calculation of Circular Cylinder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Shell II

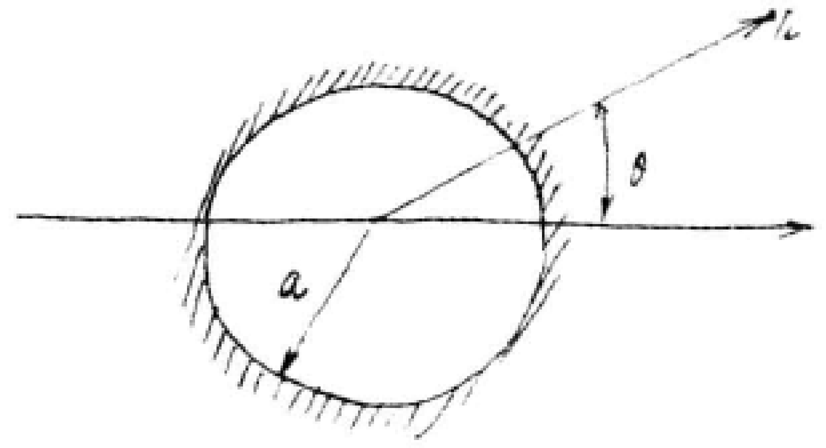
Mr. H. S. Tsien,
Guggenheim Laboratory,
California Inst. of Tech.,
Pasadena, Calif.

Preliminary Calculation of Circular Cylinder

PART (I)

The Increase in Strain Energy Due to the Presence of a Circular Hole

We have from Southwell's "Elasticity", p. 386, the following relations:



$$\widehat{rr} = \frac{1}{2}T\left(1-\frac{a^2}{r^2}\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]$$

$$\widehat{\theta\theta} = \frac{1}{2}T\left[1+\frac{a^2}{r^2}-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\widehat{r\theta} = -\frac{1}{2}T\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\widehat{rr}_0 = \frac{1}{2}T\,(1+\cos 2\theta)$$

$$\widehat{\theta\theta}_0 = \frac{1}{2}T\,(1-\cos 2\theta)$$

$$\widehat{r\theta}_0 = -\frac{1}{2}T\sin 2\theta$$

~~If we assume that $\mu=0$, i.e., vanishing Poisson's ratio,~~ then

In case of plane stress,

strain energy per unit area $= \dfrac{t}{E}\left\{\dfrac{\widehat{rr}^2+\widehat{\theta\theta}^2}{2} - \nu\,\widehat{rr}\,\widehat{\theta\theta} + (1+\nu)\,\widehat{\theta r}^2\right\}$

Thus $\left[\dfrac{\widehat{rr}^2+\widehat{\theta\theta}^2}{2} - \nu\,\widehat{\theta\theta}\,\widehat{rr} + (1+\nu)\,\widehat{\theta r}^2\right] - \left[\dfrac{\widehat{rr}_0^2+\widehat{\theta\theta}_0^2}{2} - \nu\,\widehat{\theta\theta}_0\,\widehat{rr}_0 + (1+\nu)\,\widehat{\theta r}_0^2\right]$ 263

$$= \frac{1}{4}T^2\Bigg\{\frac{1}{2}\Big[\left(1-\frac{a^2}{r^2}\right)^2\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]^2 + \left[1+\frac{a^2}{r^2} - \left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]^2$$

$$- (1+\cos 2\theta)^2 - (1-\cos 2\theta)^2\Big]$$

$$-\nu\Big[\left(1-\frac{a^2}{r^2}\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]\left[1+\frac{a^2}{r^2}-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right] - (1+\cos 2\theta)(1-\cos 2\theta)\Big]$$

$$+ (1+\nu)\left[\left(1-\frac{a^2}{r^2}\right)^2\left(1+3\frac{a^2}{r^2}\right)^2\sin^2 2\theta - \sin^2 2\theta\right]\Bigg\}$$

$$= \left(\frac{T}{2}\right)^2\Bigg\{\frac{1}{2}\Big[\frac{a^2}{r^2}\left(\frac{a^2}{r^2}-2\right)\left(1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right)^2 + 3\frac{a^2}{r^2}\cos 2\theta\left(3\frac{a^2}{r^2}\cos 2\theta - 2(1+\cos 2\theta)\right)$$

$$+ \frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\cos 2\theta\right)\left[\frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta + 2(1-\cos 2\theta)\right]\Big]$$

$$-\nu\Big[-\frac{a^2}{r^2}\left\{(1+\cos 2\theta) - 3\frac{a^2}{r^2}\cos 2\theta\right\}\left\{(1-\cos 2\theta) + \frac{a^2}{r^2}\left(1-\frac{3a^2}{r^2}\right)\cos 2\theta\right\}$$

$$+ \frac{a^2}{r^2}\cos 2\theta\left\{(1+\cos 2\theta)\left(1-\frac{3a^2}{r^2}\right) - 3(1-\cos 2\theta)\right\} - 3\left(1-\frac{3a^2}{r^2}\right)\frac{a^4}{r^4}\cos^2 2\theta\Big]$$

$$+ (1+\nu)\left[\sin^2 2\theta\left\{\left(1-\frac{a^2}{r^2}\right)\left(1+\frac{3a^2}{r^2}\right)+1\right\}\frac{a^2}{r^2}\left(2-3\frac{a^2}{r^2}\right)\right]\Bigg\} = F(r,\theta)\left(\frac{T}{2}\right)^2$$

2/4

$$F(r,\theta) = \frac{1}{2}\left\{ \frac{a^2}{r^2}\left(\frac{a^2}{r^2} - 2\right)\left[1 + \left(1 - 3\frac{a^2}{r^2}\right)\cos 2\theta\right]^2 + 3\frac{a^2}{r^2}\left[3\frac{a^2}{r^2}\cos^2 2\theta - 2\left(\cos 2\theta + \cos^2 2\theta\right)\right]\right.$$

$$+ \frac{a^2}{r^2}\left[\frac{a^2}{r^2}\left(1 - \frac{3a^2}{r^2}\right)\cos 2\theta + 2\left(1 - \cos 2\theta\right)\right] - \frac{3a^4}{r^4}\left[\frac{a^2}{r^2}\left(1 - \frac{3a^2}{r^2}\right)\cos^2 2\theta + 2\left(\cos 2\theta - \cos^2 2\theta\right)\right]$$

$$-\nu\left\{-\frac{a^2}{r^2}\left[\sin^2 2\theta - 3\frac{a^2}{r^2}\left(\cos 2\theta - \cos^2 2\theta\right) + \frac{a^2}{r^2}\left(1 - \frac{3a^2}{r^2}\right)\left(\cos 2\theta + \cos^2 2\theta\right)\right]\right.$$

$$\left. + \frac{a^2}{r^2}\left[\left(1 - \frac{3a^2}{r^2}\right)\left(\cos 2\theta + \cos^2 2\theta\right) - 3\left(\cos 2\theta - \cos^2 2\theta\right)\right] - 3\left(1 - \frac{3a^2}{r^2}\right)\frac{a^4}{r^4}\cos^2 2\theta\right\}$$

$$\left. + (1+\nu)\left\{\sin^2 2\theta\ \frac{a^2}{r^2}\left(2 - 3\frac{a^2}{r^2}\right)\left[\left(1 - \frac{a^2}{r^2}\right)\left(1 + \frac{3a^2}{r^2}\right) + 1\right]\right\}\right\}$$

Thus

$$\int_0^{2\pi} F(r,\theta)\,d\theta = \frac{\pi}{2}\left\{\frac{a^2}{r^2}\left(\frac{a^2}{r^2} - 2\right)\left[2 + \left(1 - 3\frac{a^2}{r^2}\right)^2\right] + 3\frac{a^2}{r^2}\left[3\frac{a^2}{r^2} - 2\right]\right.$$

$$\left. + \frac{a^2}{r^2}\cdot 4 - \frac{3a^4}{r^4}\left[\frac{a^2}{r^2}\left(1 - \frac{3a^2}{r^2}\right) - 2\right]\right\}$$

$$-\nu\pi\left\{-\frac{a^2}{r^2}\left[1 + 3\frac{a^2}{r^2} + \frac{a^2}{r^2}\left(1 - \frac{3a^2}{r^2}\right)\right] + \frac{a^2}{r^2}\left[\left(1 - \frac{3a^2}{r^2}\right) + 3\right] - 3\left(1 - \frac{3a^2}{r^2}\right)\frac{a^4}{r^4}\right\}$$

$$+ \pi(1+\nu)\ \frac{a^2}{r^2}\left(2 - 3\frac{a^2}{r^2}\right)\left[2 + 2\frac{a^2}{r^2} - \frac{3a^4}{r^4}\right]\Big\}$$

Putting $\frac{a}{r} = \frac{1}{R}$

$$= \frac{\pi}{2}\left\{\frac{1}{R^2}\left(\frac{1}{R^2} - 2\right)\left(3 - 6\frac{1}{R^2} + 9\frac{1}{R^4}\right) + 3\frac{1}{R^2}\left(3\frac{1}{R^2} - 2\right) + 4\frac{1}{R^2} - 3\frac{1}{R^4}\left[\frac{1}{R^2} - \frac{3}{R^4} - 2\right]\right\}$$

$$-\nu\pi\left\{-\frac{1}{R^2}\left[1 + 4\frac{1}{R^2} - \frac{3}{R^4}\right] + \frac{1}{R^2}\left[4 - \frac{3}{R^2}\right] - 3\frac{1}{R^4}\left(1 - \frac{3}{R^2}\right)\right\}$$

$$+ \pi(1+\nu)\frac{1}{R^2}\left(2 - 3\frac{1}{R^2}\right)\left(2 + \frac{2}{R^2} - \frac{3}{R^4}\right)\Big\}$$

If for simplicity $\theta = 0$, then

$$\frac{R^4}{\pi}\int_0^{2\pi} F(r,\theta)\,d\theta = \Big[\tfrac{1}{2}\Big(3 - \frac{6}{R^2} + 12 + \frac{9}{R^4} - 18 + 9 - \frac{3}{R^2} + \frac{9}{R^4} + 6\Big)$$

$$+ \Big(-6 + 4 - \frac{6}{R^2} - \frac{6}{R^2} + \frac{9}{R^4}\Big)\Big]$$

$$= \Big[\,4 - \frac{33}{2}\frac{1}{R^2} + 18\frac{1}{R^4}\Big]$$

Energy increase due to the presence of hole of radius "a"

$$= \int_a^{\infty}\int_0^{2\pi} F(r,\theta)\Big(\frac{\sigma}{2}\Big)^2 \frac{t}{E}\, r\,dr\,d\theta$$

$$= a^2\frac{t}{E}\Big(\frac{\sigma}{2}\Big)^2 \int_1^{\infty}\int_0^{2\pi} F(r,\theta)\,d\theta \cdot R\,dR$$

$$= \frac{a^2\pi t}{E}\Big(\frac{\sigma}{2}\Big)^2 \int_1^{\infty}\Big[\frac{4}{R^3} - \frac{33}{2}\frac{1}{R^5} + \frac{18}{R^7}\Big]dR$$

$$= \frac{a^2\pi t}{E}\Big(\frac{\sigma}{2}\Big)^2\Big[\,2 - \frac{33}{8} + 3\Big] = \boxed{\frac{a^2\pi t}{E}\Big(\frac{\sigma}{2}\Big)^2\frac{7}{8} = \frac{a^2\pi t\sigma^2}{E}\frac{7}{32}}$$

$$\frac{\partial u}{\partial r} = \frac{1}{E}\,\widehat{rr}$$

$$\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}\,\widehat{\theta\theta}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{2}{E}\,\widehat{r\theta}$$

$$\frac{\partial u}{\partial r} = \frac{\sigma}{2E}\left(1 - \frac{a^2}{r^2}\right)\left[1 + \left(1 - \frac{3a^2}{r^2}\right)\cos 2\theta\right]$$

$$= \frac{\sigma}{2E}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$u = \frac{\sigma}{2E}\left[\left(r + \frac{a^2}{r}\right) + \left(r + \frac{4a^2}{r} - \frac{a^4}{r^3}\right)\cos 2\theta + F(\theta)\right]$$

$$\frac{\partial v}{\partial \theta} = \frac{1}{E}\,r \cdot \widehat{\theta\theta} - u$$

$$= \frac{\sigma}{2E}\left\{ r + \frac{a^2}{r} - \left(r + \frac{3a^4}{r^3}\right)\cos 2\theta - \left(r + \frac{a^2}{r}\right) - \left(r + \frac{4a^2}{r} - \frac{a^4}{r^3}\right)\cos 2\theta - F(\theta)\right\}$$

$$= \frac{\sigma}{2E}\left\{ -2\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\cos 2\theta - F(\theta)\right\}$$

$$v = \frac{\sigma}{2E}\left\{ -\left(r + \frac{2a^2}{r} + \frac{a^4}{r^3}\right)\sin 2\theta - \int F(\theta) + G(r)\right\}$$

Check.

$$\frac{\sigma}{2E}\Big[-2\left(1+\frac{4a^2}{r^2}-\frac{a^4}{r^4}\right)\sin 2\theta+\frac{F'(\theta)}{r}-\left(1-\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta+G'(r)$$

$$+\left(1+\frac{2a^2}{r^2}+\frac{a^4}{r^4}\right)\sin 2\theta+\frac{\int F(\theta)}{r}+\frac{G(r)}{r}\Big]$$

$$=\frac{\sigma}{E}\Big[\left(\frac{2a^2}{r^2}+\frac{2a^4}{r^4}-1-\frac{4a^2}{r^2}+\frac{a^4}{r^4}\right)\sin 2\theta+\frac{F'(\theta)+G(r)+\int F(\theta)}{r}+G'(r)\Big]$$

$$=-\frac{\sigma}{E}\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta+\frac{\sigma}{E}\Big[\quad\Big]$$

$$\therefore\qquad \frac{v}{r}=-\frac{\sigma}{2E}\left(1+\frac{a^2}{r^2}\right)^2\sin 2\theta$$

$$\frac{u}{r}=\frac{\sigma}{2E}\Big[\left(1+\frac{a^2}{r^2}\right)+\left(1+\frac{4a^2}{r^2}-\frac{a^4}{r^4}\right)\cos 2\theta\Big]$$

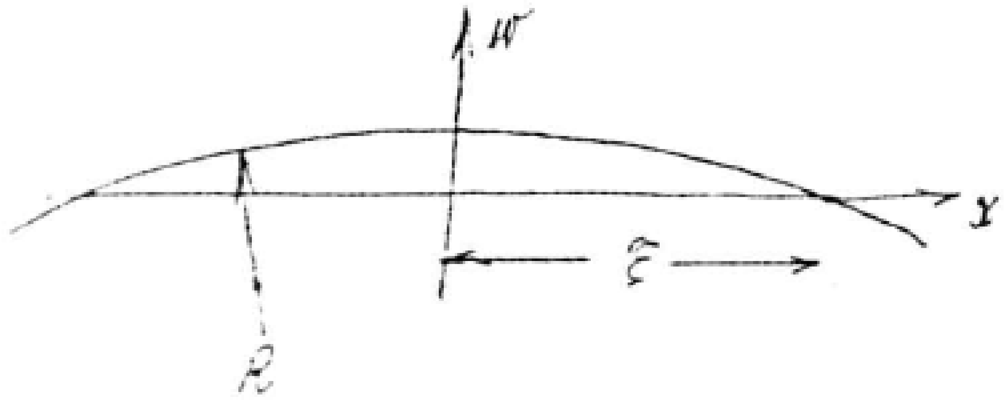
At $r=a$,

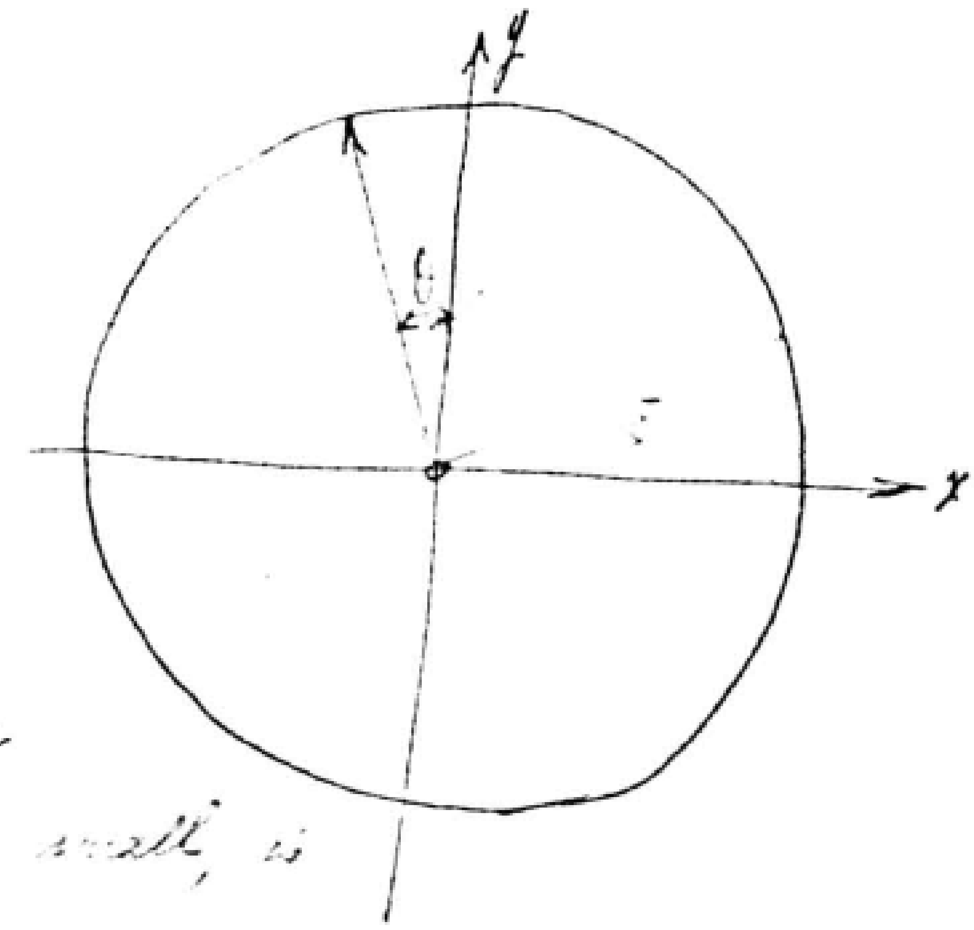
$$\boxed{\begin{aligned}\frac{v}{a}&=-\frac{\sigma}{2E}\,4\sin 2\theta=-\frac{2\sigma}{E}\sin 2\theta\\ \frac{u}{a}&=\frac{\sigma}{2E}\left(2+4\cos 2\theta\right)=\frac{\sigma}{E}\left(1+2\cos 2\theta\right)\end{aligned}}$$

PART (II)

Calculation of the Strain Energy of a Cylindrical Shell of Circular Plane Form with Given Displacement

1) The vertical displacement:





The original form of the shell, where the angular extension of the shell is small, is

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{\xi}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2} = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2}$$

After Buckling

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{\xi}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2} - f\left[\frac{\left(\frac{\xi\sqrt{\xi^2-y^2}}{\xi R}\right)^2 - \left(\frac{x}{R}\right)^2}{2}\right]$$

$$\left(\frac{w}{R}\right) = \frac{1}{2}\left\{\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{x}{R}\right)^2}{2} - f\left[\left(\frac{a}{R}\right)^2 - \left\{\left(\frac{y}{R}\right)^2 + \left(\frac{x}{R}\right)^2\right\}\right]\right\}$$

Changing to polar coördinates

$$\left.\begin{aligned} \left(\frac{w}{R}\right)_0 &= \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} \\ \left(\frac{w}{R}\right) &= \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left[\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r^2}{R^2}\right)}{2}\right] \end{aligned}\right\} \text{Form (I)}$$

To avoid any bending moments at the edges, we approximate the circular arc with a sine wave. The amplitude of the sine wave is given by

$$R(1-\cos\beta) = 2R\sin^2\frac{\beta}{2} = R\,\frac{\beta^2}{2} \cong R\,\frac{\left(\frac{a}{R}\right)^2}{2} \cong \underline{\underline{\frac{a^2}{2R}}}$$

$$\cancel{\left(\frac{w}{R}\right)_0 = \frac{a^2}{2R}\cos\frac{\pi x}{2a}}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\sin^2\theta}{2} - f\cos\frac{\pi r}{2a}$$

---

2) The equation of equilibrium of stresses in the middle plane:

$$\left.\begin{aligned} &\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \\ &\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0 \end{aligned}\right\}$$

These equations are satisfied by introducing the stress function φ,

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\varphi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\varphi}{\partial r\,\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$\sigma_r = E\left\{ \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2 \right\}$$

$$\sigma_\theta = E\left\{ \frac{u}{r} + \frac{\partial v}{r\,\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{2r^2}\left(\frac{\partial w_0}{\partial \theta}\right)^2 \right\}$$

$$\tau_{r\theta} = \frac{E}{2}\left\{ \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial w_0}{\partial \theta} \right\}$$

$$\frac{\partial^2 w}{\partial x^2} = \cos^2\theta \frac{\partial^2 w}{\partial r^2} - \frac{\sin 2\theta}{r}\frac{\partial^2 w}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial w}{\partial r} + \frac{\sin 2\theta}{r^2}\frac{\partial w}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

$$\frac{\partial^2 w}{\partial y^2} = \sin^2\theta \frac{\partial^2 w}{\partial r^2} + \frac{\sin 2\theta}{r}\frac{\partial^2 w}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial w}{\partial r} - \frac{\sin 2\theta}{r^2}\frac{\partial w}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial \phi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \phi}{\partial \theta}\sin\theta\right)$$

$$= \sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^2}\sin^2\theta \frac{\partial \phi}{\partial \theta} - \frac{\sin^2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial \phi}{\partial r}$$

$$- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\cos^2\theta}{r^2}\frac{\partial \phi}{\partial \theta}$$

$$= \sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r^2} - \frac{\cos 2\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\cos 2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial \phi}{\partial r}$$

Thus

$$- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2 = \sin^2\theta \cos^2\theta \left(\frac{\partial^2 \omega}{\partial r^2}\right)^2 + \frac{\cos^2 2\theta}{r^4}\left(\frac{\partial \varphi}{\partial \theta}\right)^2 + \frac{\cos^2 2\theta}{r^2}\left(\frac{\partial^2 \varphi}{\partial r \partial \theta}\right)^2$$

$$+ \frac{\sin^2\theta \cos^2\theta}{r^2}\left(\frac{\partial \omega}{\partial r}\right)^2 + \frac{\sin^2\theta \cos^2\theta}{r^4}\left(\frac{\partial^2 \varphi}{\partial \theta^2}\right)^2 - \frac{\sin 2\theta \cos 2\theta}{r^2}\frac{\partial \omega}{\partial \theta}\frac{\partial^2 \omega}{\partial r^2}$$

$$+ \frac{\sin 2\theta \cos 2\theta}{r}\frac{\partial^2 \omega}{\partial r^2}\frac{\partial^2 \omega}{\partial r \partial \theta} - \frac{1}{2}\frac{\sin^2 2\theta}{r}\frac{\partial \omega}{\partial r}\frac{\partial^2 \varphi}{\partial r^2} - \frac{1}{2}\frac{\sin^2 2\theta}{r^2}\frac{\partial^2 \varphi}{\partial r^2}\frac{\partial^2 \omega}{\partial \theta^2}$$

$$- 2\frac{\cos^2 2\theta}{r^3}\frac{\partial \omega}{\partial \theta}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{\sin 2\theta \cos 2\theta}{r^3}\frac{\partial \varphi}{\partial r}\frac{\partial \omega}{\partial \theta} + \frac{\sin 2\theta \cos 2\theta}{r^4}\frac{\partial \varphi}{\partial \theta}\frac{\partial^2 \varphi}{\partial \theta^2}$$

$$- \frac{\sin 2\theta \cos 2\theta}{r^2}\frac{\partial \omega}{\partial r}\frac{\partial^2 \omega}{\partial r \partial \theta} - \frac{\sin 2\theta \cos 2\theta}{r^3}\frac{\partial^2 \omega}{\partial r \partial \theta}\frac{\partial^2 \omega}{\partial \theta^2} + \frac{1}{2}\frac{\sin^2 2\theta}{r^3}\frac{\partial \omega}{\partial r}\frac{\partial^2 \omega}{\partial \theta^2}$$

$$\boxed{\begin{aligned}\left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2 - \frac{\partial^2 \omega}{\partial x^2}\frac{\partial^2 \omega}{\partial y^2} = {} & \frac{1}{r^4}\left(\frac{\partial \omega}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial^2 \omega}{\partial r \partial \theta}\right)^2 - \frac{1}{r}\frac{\partial \omega}{\partial r}\frac{\partial^2 \omega}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 \omega}{\partial r^2}\frac{\partial^2 \omega}{\partial \theta^2} \\ & - 2\frac{1}{r^3}\frac{\partial \omega}{\partial \theta}\frac{\partial^2 \omega}{\partial r \partial \theta}\end{aligned}}$$

We have $\left(\frac{w}{R}\right)_0 = \dfrac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left\{\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2}{2}\right\}$$

Thus

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial\theta} = -\left(\frac{r}{R}\right)^2 \sin\theta\cos\theta = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial w_0}{\partial\theta} = -\left(\frac{r}{R}\right)^2 \sin\theta\cos\theta = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r\partial\theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r\partial\theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial\theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial\theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta + f\frac{1}{r}\left(\frac{r}{R}\right)^2 \\ \frac{1}{R}\frac{\partial w_0}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta + \frac{f}{R^2} \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta \end{cases}$$

$$\frac{1}{r^4}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{r^4}\left(\frac{\partial w_0}{\partial \theta}\right)^2 = 0$$

$$\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r \partial \theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 w_0}{\partial r \partial \theta}\right)^2 = 0$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\} = \left(\frac{1}{R^2}\sin^2\theta\right)^2 - \left[\frac{1}{R^2}\sin^2\theta - \frac{f}{R^2}\right]^2$$

$$= \frac{1}{R^4}\left[2\sin^2\theta - f\right]f \cdot R^2$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\} = \frac{1}{R^2}\cos 2\theta\left\{\frac{f}{R^2}\right\} = \frac{1}{R^4} f \cos 2\theta \cdot R^2$$

$$-2\left\{\frac{1}{r^3}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial r \partial \theta}\cdots\right\} = \frac{1}{R^2}\sin 2\theta\left\{\cdot\ \cdot\right\} = 0$$

The equation for the stress function φ is simply

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = \frac{Ef}{R^2}\left\{\cos 2\theta + 2\sin^2\theta - f\right\}$$

$$= \frac{Ef(1-f)}{R^2} = K$$

First we have to find the particular integral of the equation

$$\underbrace{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = K}$$

Assuming φ independent of θ,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) = K$$

Let $\quad \phi = c r^p$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = [p(p-1) + p] r^{p-2} = c p^2 r^{p-2}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) = c p^2 (p-2)^2 r^{p-4} = K$$

$$\therefore \quad \underline{\underline{p = 4}}$$

Also $$\phi = c r^p$$

$$c \cdot p^2 (p-2)^2 = K \qquad c = \frac{K}{16 \cdot 4} = \underline{\underline{\frac{K}{64}}}$$

Hence the particular integral is

$$\boxed{\phi = \frac{K}{64} r^4}$$

Due to the symmetry of this problem, the solution of the homogeneous equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0$$

can be written as

$$\phi = \sum_{n=1}^{\infty} r^{2n}\left(A_n \cos 2(n-1)\theta + B_n \cos 2n\theta \right) \quad \cdots [illegible]$$

Therefore the complete solution is

$$\phi = \frac{K}{64} r^4 + \sum_{n=1}^{\infty} r^{2n} \left[A_n \cos 2(n-1)\theta + B_n \cos 2n\theta\right] \quad \cdots [illegible]$$

Thus

$$\sigma_r = \frac{K}{16}r^2 + \sum_1 r^{2(n-1)}\left[A_n\left\{2n-4(n-1)^2\right\}\cos 2(n-1)\theta + B_n\left\{2n-4n^2\right\}\cos 2n\theta\right]$$

$$\sigma_\theta = \frac{3K}{16}r^2 + \sum_1 2n(2n-1)r^{2(n-1)}\left[A_n \cos 2(n-1)\theta + B_n \cos 2n\theta\right]$$

$$\tau_{r\theta} = \sum_1 (2n-1)r^{2(n-1)}\left[2(n-1)A_n \sin 2(n-1)\theta + 2n B_n \sin 2n\theta\right]$$

The boundary conditions are at $r=a$, $\sigma_r = \tau_{r\theta} = 0$, $\sigma_\theta = \sigma(1-2\cos 2\theta)$

Thus

$$0 = \frac{K}{16}a^2 + \sum_1\left[\left\{2n-4(n-1)^2\right\}a_n \cos 2(n-1)\theta + \left\{2n-4n^2\right\}b_n \cos 2n\theta\right]$$

$$\sigma(1-2\cos 2\theta) = \frac{3K}{16}a^2 + \sum_1 2n(2n-1)\left[a_n \cos 2(n-1)\theta + b_n \cos 2n\theta\right]$$

$$0 = \sum_1 (2n-1)\left[2(n-1)a_n \sin 2(n-1)\theta + 2n b_n \sin 2n\theta\right]$$

where $\underline{a_n = a^{2(n-1)}A_n}$

$$\therefore \quad \frac{Ef(1-f)}{16}\left(\frac{a}{R}\right)^2 = -2a_1, \quad 0 = \left\{2(n+1)-4n^2\right\}a_{n+1} + \left\{2n-4n^2\right\}b_n$$

$$\sigma - \frac{3Ef(1-f)}{16}\left(\frac{a}{R}\right)^2 = 2a_1, \quad -2\sigma = 2b_1 + 12a_2,$$

$$2(n+1)\left[2n+1\right]a_{n+1} + 2n(2n-1)b_n = 0$$

$$(2n+1)2n\, a_{n+1} + 2n(2n-1)\, b_n = 0$$

It is thus impossible to satisfy the stress boundary condition.

But
$$\sigma_r = E\frac{\partial u}{\partial r} + \frac{E}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \frac{\partial w_0}{\partial r}^2\right\}$$

$$= E\frac{\partial u}{\partial r} + \frac{E}{2}f\left(\frac{r}{R}\right)\left[f\left(\frac{r}{R}\right) - 2\left(\frac{r}{R}\right)\sin^2\theta\right]$$

$$= E\frac{\partial u}{\partial r} + \frac{E}{2}f(f-1)\left(\frac{r}{R}\right)^2 + \frac{E}{2}f\left(\frac{r}{R}\right)^2\cos 2\theta$$

$$\sigma_\theta = E\left\{\frac{u}{r} + \frac{\partial v}{r\partial\theta}\right\}$$

$$\tau_{r\theta} = \frac{E}{2}\left\{\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right\} - \frac{E}{2}\frac{1}{2}\left(\frac{r}{R}\right)\sin 2\theta \cdot f\left(\frac{r}{R}\right)$$

Hence
$$E\frac{\partial u}{\partial r} = \frac{9}{16}Ef(1-f)\left(\frac{r}{R}\right)^2 - \frac{1}{2}Ef\left(\frac{r}{R}\right)^2\cos 2\theta$$
$$+ \sum_1 \left(\frac{r}{R}\right)^{2(n-1)}\Big[A_n\left\{2n - 4(n-1)^2\right\}\cos 2(n-1)\theta$$
$$+ B_n\left\{2n - 4n^2\right\}\cos 2n\theta\Big]$$

$$E\left[\frac{u}{r} + \frac{\partial v}{r\partial\theta}\right] = \frac{3Ef(1-f)}{16}\left(\frac{r}{R}\right)^2 + \sum_1 2n(2n-1)\left(\frac{r}{R}\right)^{2(n-1)}\Big[A_n\cos 2(n-1)\theta$$
$$+ B_n\cos 2n\theta\Big]$$

$$E\left[\frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right] = \frac{1}{2}f\left(\frac{r}{R}\right)^2\sin 2\theta$$
$$+ \sum_1 2(2n-1)\left(\frac{r}{R}\right)^{2(n-1)}\Big[2(n-1)A_n\sin 2(n-1)\theta$$
$$+ 2nB_n\sin 2n\theta\Big]$$

$$E\frac{u}{R} = \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^3 - \frac{1}{6}Ef\left(\frac{a}{R}\right)^3\cos 2\theta$$

$$+ \sum \frac{1}{2n-1}\left(\frac{a}{R}\right)^{2n-1}\left[A_n\left\{2n-4(n-1)^2\right\}\cos 2(n-1)\theta + B_n\left\{2n-4n^2\right\}\cos 2n\theta\right]$$

$$\boxed{\begin{aligned} E\frac{u}{\rho} &= \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^3 - \frac{1}{6}Ef\left(\frac{a}{R}\right)^3\cos 2\theta \\ &+ \sum\left(\frac{a}{R}\right)^{2n-1}\left[A_n(4-2n)\cos 2(n-1)\theta - B_n\, 2n\cos 2n\theta\right]\end{aligned}}$$

$$E\frac{u}{r} = \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^2 - \frac{1}{6}Ef\left(\frac{a}{R}\right)^2\cos 2\theta$$

$$+ \sum\left(\frac{a}{R}\right)^{2(n-1)}\left[(4-2n)A_n\cos 2(n-1)\theta - 2nB_n\cos 2n\theta\right]$$

$$E\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{6}Ef\left(\frac{a}{R}\right)^2\cos 2\theta + \sum\left(\frac{a}{R}\right)^{2(n-1)}\left[4(n^2-1)A_n\cos 2(n-1)\theta + 4n^2B_n\cos 2n\theta\right]$$

$$\boxed{\begin{aligned} E\frac{v}{R} = \frac{1}{12}Ef\left(\frac{a}{R}\right)^3\sin 2\theta + \sum\left(\frac{a}{R}\right)^{2n-1}\Big[&-2(n+1)A_n\sin 2(n-1)\theta \\ &+ 2nB_n\sin 2n\theta\Big]\end{aligned}}$$

But from page 267

$$E\frac{u}{R} = \sigma\left(\frac{a}{R}\right)\left[1+2\cos 2\theta\right]$$

$$E\frac{v}{R} = -\sigma\left(\frac{a}{R}\right)\sin 2\theta$$

$$\left.\begin{aligned} \sigma \frac{a}{R} &= \frac{3}{16} E f(1-f)\left(\frac{a}{R}\right)^3 + \left(\frac{a}{R}\right)\cdot 2A_1 \\ 2\sigma\left(\frac{a}{R}\right) &= -\frac{1}{6} E f \left(\frac{a}{R}\right)^3 - 2\left(\frac{a}{R}\right) B_1 \\ -\sigma\left(\frac{a}{R}\right) &= \frac{1}{12} E f\left(\frac{a}{R}\right)^3 + 2\left(\frac{a}{R}\right) B_1 \end{aligned}\right\}$$

Three equations for three unknowns f, A_1, B_1

$$\frac{\sigma}{E} = -\frac{1}{12} f \left(\frac{a}{R}\right)^2 \qquad \therefore \quad \boxed{f = \frac{-12\sigma}{E\left(\frac{a}{R}\right)^2}}$$

$$b_1 = -\frac{1}{6} E f \left(\frac{a}{R}\right)^2 - \sigma$$

$$= 2\sigma - \sigma = \sigma. \qquad \boxed{B_1 = \sigma}$$

$$\therefore \quad \sigma = \frac{3}{16} E f (1-f)\left(\frac{a}{R}\right)^2 + 2A_1$$

$$A_1 = \frac{\sigma}{2} - \frac{3}{32}(-12\sigma)\left[1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right]$$

$$\boxed{A_1 = \frac{\sigma}{2} + \frac{9}{8}\sigma\left[1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right]}$$

$$\boxed{\begin{aligned}\left(\frac{\phi}{R^2}\right) &= \underbrace{\frac{Ef(1-f)}{8t}}_{C_1}\left(\frac{r}{R}\right)^4 + \left(\frac{r}{R}\right)^2 \underbrace{\left[\frac{\sigma}{2} - \frac{3}{32} E f(1-f)\left(\frac{a}{R}\right)^2\right]}_{C_2} \\ &+ \left(\frac{r}{R}\right)^2 \underbrace{\sigma}_{C_3} \cos 2\theta\end{aligned}}$$

or $\quad \dfrac{\phi}{R^2} = C_1\left(\dfrac{r}{R}\right)^4 + C_2\left(\dfrac{r}{R}\right)^2 + C_3\left(\dfrac{r}{R}\right)^2 \cos 2\theta$

$$\sigma_r = 4C_1\left(\frac{r}{R}\right)^2 + 2C_2 + 2C_3 \cos 2\theta - 4C_3 \cos 2\theta$$

$$\sigma_\theta = 12C_1\left(\frac{r}{R}\right)^2 + 2C_2 + 2C_3 \cos 2\theta$$

$$\tau_{r\theta} = + 2C_3 \sin 2\theta$$

The extensional energy

$$4\,\frac{Et}{2}\,\frac{1}{E^2}\,R^2 \int_0^{a/R}\int_0^{2\pi} \Big\{ \left[2C_1\left(\frac{r}{R}\right)^2 + C_2 - C_3 \cos 2\theta\right]^2$$

$$+ \left[6C_1\left(\frac{r}{R}\right)^2 + C_2 + C_3 \cos 2\theta\right]^2 + 2C_3^2 \sin^2 2\theta \Big\} \left(\frac{r}{R}\right) d\left(\frac{r}{R}\right) d\theta$$

$$= \frac{2tR^2\pi}{E}\int_0^{a/R}\left[8C_1^2\left(\frac{r}{R}\right)^4 + 2C_2^2 + C_3^2 + 72C_1^2\left(\frac{r}{R}\right)^4 + 2C_2^2 + C_3^2 + 2C_3^2\right]\left(\frac{r}{R}\right) d\left(\frac{r}{R}\right)$$

$$= \frac{2tR^2\pi}{E}\int_0^{a/R}\left[80C_1^2\left(\frac{r}{R}\right)^5 + 4\left(C_2^2 + C_3^2\right)\frac{r}{R}\right] d\left(\frac{r}{R}\right)$$

$$= \frac{8tR^2\pi}{E}\left\{ \frac{10}{3} C_1^2 \left(\frac{a}{R}\right)^6 + \frac{1}{2}(C_2^2 + C_3^2)\left(\frac{a}{R}\right)^2 \right\}$$

The bending energy

$$\frac{1}{R}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] - \frac{1}{R}\left[\frac{\partial^2 w_0}{\partial R^2} + \frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{\mu}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right]$$

$$= \frac{1}{R^2}\Big[$$

In case of polar coordinates

$$K_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2}$$

$$K_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$\tau = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w_0}{\partial \theta}\right)$$

$$\frac{1}{R}K_1 = \frac{f}{R^2}$$

$$\tau = 0$$

$$\frac{1}{R}K_2 = \frac{f}{R^2}$$

$$\text{Bending energy} = \frac{Et}{2}\frac{t^2}{12}\, 2\left(\frac{f}{R}\right)^2 \pi a^2 = \frac{E}{12}\left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 f^2 R^3$$

Total energy

$$\frac{W}{R^3} = \frac{\delta\left(\frac{t}{R}\right)}{E}\pi\left(\frac{a}{R}\right)^2\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4+\frac{1}{2}(C_2^2+C_3^2)\right\}+\frac{E}{12}\left(\frac{t}{R}\right)^3 f^2\,\pi\left(\frac{a}{R}\right)^2$$

$$=\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{\delta}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4+\frac{1}{2}(C_2^2+C_3^2)\right\}+\frac{E}{12}\left(\frac{t}{R}\right)^2 f^2\right]$$

Decrease in energy

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{\sigma^2}{2E}-\frac{\delta}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4+\frac{1}{2}(C_2^2+C_3^2)\right\}-\frac{E}{12}\left(\frac{t}{R}\right)^2 f^2\right]$$

Total decrease in energy

$$\frac{\Delta W}{R^3} = \frac{t}{R}\pi\left(\frac{a}{R}\right)^2\left[\frac{9}{32}\frac{\sigma^2}{E}-\frac{\delta}{E}\left\{\frac{10}{3}C_1^2\left(\frac{a}{R}\right)^4+\frac{1}{2}(C_2^2+C_3^2)\right\}-\frac{E}{12}\left(\frac{t}{R}\right)^2 f^2\right]$$

Or write out completely, we have

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\left[\frac{9}{32}\frac{\sigma^2}{E}-\frac{\delta}{E}\left\{\frac{15}{128}\sigma^2\left[1+\frac{120\sigma}{E\left(\frac{a}{R}\right)^2}\right]^2+\frac{1}{2}\left[\sigma^2\right.\right.\right.$$

$$\left.\left.\left.+\left\{\frac{\sigma}{2}+\frac{9}{8}\sigma\left(1+\frac{120\sigma}{E\left(\frac{a}{R}\right)^2}\right)\right\}^2\right]\right\}-\frac{E}{12}\left(\frac{t}{R}\right)^2\frac{144\sigma^2}{E^2\left(\frac{a}{R}\right)^4}\right]$$

$$=\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{E}\left[\frac{9}{32}-\frac{15}{16}\left\{1+\frac{120\sigma}{E\left(\frac{a}{R}\right)^2}\right\}^2-5-\frac{81}{16}\left\{1+\frac{120\sigma}{E\left(\frac{a}{R}\right)^2}\right\}^2\right.$$

$$\left.-\frac{9}{2}\left\{1+\frac{120\sigma}{E\left(\frac{a}{R}\right)^2}\right\}-12\left(\frac{t}{R}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4}\right]$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{E}\left[\frac{9}{32} - 5 - 6\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\}^2 - \frac{9}{2}\left\{1 + \frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right\} - 12\left(\frac{t}{R}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4}\right]$$

$$= \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{E}\left[-\frac{487}{32} - 198\frac{\sigma}{E\left(\frac{a}{R}\right)^2} - 864\left(\frac{\sigma}{E}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4} - 12\left(\frac{t}{R}\right)^2\frac{1}{\left(\frac{a}{R}\right)^4}\right]$$

Putting $\sigma = -\sigma$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left[198\left(\frac{\sigma}{E}\right) - \frac{487}{32}\left(\frac{a}{R}\right)^2 - 864\left(\frac{\sigma}{E}\right)^2\frac{1}{\left(\frac{a}{R}\right)^2} - 12\left(\frac{\sigma}{E}\right)^2\left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\frac{1}{\left(\frac{a}{R}\right)^2}\right]$$

$$\frac{487}{32}\left(\frac{a}{R}\right)^4 = 12\left(\frac{\sigma}{E}\right)^2\left[72 + \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\right]$$

$$\therefore \quad \left(\frac{a}{R}\right)^2 = \sqrt{\frac{384}{487}}\,\frac{\sigma}{E}\left[72 + \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\right]^{\frac{1}{2}}$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left[198\left(\frac{\sigma}{E}\right) - 2\sqrt{\frac{487\times 12}{32}}\left(\frac{\sigma}{E}\right)\left\{72 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2\right\}^{\frac{1}{2}}\right]$$

$$= \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left(\frac{\sigma}{E}\right)\left[198 - 2\sqrt{\frac{1461}{8}\left\{72 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2\right\}}\right]$$

Thus at σ_c, $\Delta W = 0$

$$198^2 = \frac{1461}{8}\left\{72 + \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right)^2\right\}$$

283

$$\sqrt{198^2 \frac{8}{1461} - 66} = \left(\frac{E\left(\frac{t}{R}\right)}{\sigma}\right) = 12.19$$

100

300000

$$\boxed{\sigma_{cri} = 0.0821\, E\left(\frac{t}{R}\right)}$$

$$\begin{array}{r} 214.66 \\ 66 \\ \hline 148.66 \end{array}$$

at σ_{cri}

$$198\sqrt{\frac{32}{487 \times 12}} = \sqrt{66 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2}$$

$$\left(\frac{a}{R}\right)^2 = \frac{32}{487}\,\frac{\sigma}{E} = \frac{32}{487} \times 0.0821\left(\frac{t}{R}\right)$$

$$\boxed{\left(\frac{a}{R}\right) = 0.0735\sqrt{\left(\frac{t}{R}\right)}}$$

The residual stress at the edges are

$$\sigma_r = 4C_1\left(\frac{a}{R}\right)^2 + 2C_2 - 2C_3 \cos 2\theta$$

$$\boxed{\sigma_r = \frac{Ef(1-f)}{16}\left(\frac{a}{R}\right)^2 + \left\{\sigma - \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^2\right\} - 2\sigma\cos 2\theta}$$

$$\boxed{\sigma_\theta = \frac{3Ef(1-f)}{16}\left(\frac{a}{R}\right)^2 + \left\{\sigma - \frac{3}{16}Ef(1-f)\left(\frac{a}{R}\right)^2\right\} + 2\sigma\cos 2\theta}$$

$$\boxed{\tau_{r\theta} = 2\sigma\sin 2\theta}$$

The additional stress system is

$$\begin{cases} \sigma_r = -\frac{Ef(1-f)}{8}\left(\frac{a}{R}\right)^2 + \sigma(1-2\cos 2\theta) \\ \sigma_\theta = 4\sigma\cos 2\theta \\ \tau_{r\theta} = 2\sigma\sin 2\theta \end{cases}$$

[See p. 385, Southwell's "elasticity", Eq. (43) & (45)

$$R_0 = \sigma - \frac{Ef(1-f)}{8}\left(\frac{a}{R}\right)^2$$

$$\left.\begin{aligned} 2\sigma &= 6Q_2 + 4S_2 \\ -2\sigma &= 6Q_2 + 2S_2 \end{aligned}\right\} \qquad 4\sigma = 2S_2 \qquad \begin{aligned} S_2 &= 2\sigma \\ Q_2 &= -\sigma \end{aligned}$$

Notice above that only the σ_r & $\tau_{r\theta}$ conditions are satisfied 245

$$\sigma_r = \left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{r}\right)^2 + \left\{6\sigma\left(\frac{a}{r}\right)^4 - 8\sigma\left(\frac{a}{r}\right)^2\right\}\cos 2\theta$$

$$\sigma_\theta = -\left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{r}\right)^2 - 6\sigma\left(\frac{a}{r}\right)^4 \cos 2\theta$$

$$\tau_{r\theta} = \left\{6\sigma\left(\frac{a}{r}\right)^4 - 4\sigma\left(\frac{a}{r}\right)^2\right\}\sin 2\theta$$

$$\pi a^2 \frac{t}{2E}\int_1^\infty \left[2\left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}^2 \frac{1}{\left(\frac{r}{a}\right)^3} + \left\{6\sigma\frac{1}{\left(\frac{r}{a}\right)^4} - 8\sigma\frac{1}{\left(\frac{r}{a}\right)^2}\right\}^2\left(\frac{r}{a}\right)\right.$$

$$+ 2\left\{\sigma - \frac{E f(1-f)}{8}\left(\frac{a}{R}\right)^2\right\}^2 \frac{1}{\left(\frac{r}{a}\right)^3} + 36\sigma^2\frac{1}{\left(\frac{r}{a}\right)^7}$$

$$\left. + 2\left\{6\sigma\frac{1}{\left(\frac{r}{a}\right)^4} - 4\sigma\frac{1}{\left(\frac{r}{a}\right)^2}\right\}^2\left(\frac{r}{a}\right)\right] d\left(\frac{r}{a}\right)$$

$$= \frac{\pi a^2 t\sigma^2}{2E}\left[2\left\{1 - \frac{E f(1-f)}{\sigma 8}\left(\frac{a}{R}\right)^2\right\}^2 + (6 - 24 + 32 + 6 + 12 - 24 + 16)\right]$$

$$= \frac{\pi a^2 t\sigma^2}{E}\left\{\left[1 - \frac{E f(1-f)}{8\sigma}\left(\frac{a}{R}\right)^2\right]^2 + 12\right\}$$

46

$$\frac{\Delta W_1}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{13\left(\frac{a}{R}\right)^2 - \frac{Ef(1-f)}{4\sigma}\left(\frac{a}{R}\right)^4 + \frac{E^2f^2(1-f)^2}{64\sigma^2}\left(\frac{a}{R}\right)^6\right\}$$

$$= \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{13\left(\frac{a}{R}\right)^2 + 3\left(1+\frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right)\left(\frac{a}{R}\right)^4\right.$$

$$\left. + \frac{9}{4}\left(1+\frac{12\sigma}{E\left(\frac{a}{R}\right)^2}\right)^2\left(\frac{a}{R}\right)^6\right\}$$

$$= \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{13\left(\frac{a}{R}\right)^2 + 3\left(\frac{a}{R}\right)^4 + \frac{9}{4}\left(\frac{a}{R}\right)^6 + \frac{36\sigma}{E}\left(\frac{a}{R}\right)^2\right.$$

$$\left. + \frac{54\sigma}{E}\left(\frac{a}{R}\right)^4 + 9\times 36\left(\frac{\sigma}{E}\right)^2\left(\frac{a}{R}\right)^2\right\}$$

$$\frac{\Delta W_1}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{\left[13 - 36\left(\frac{\sigma}{E}\right) + 324\left(\frac{\sigma}{E}\right)^2\right]\left(\frac{a}{R}\right)^2 + \left[3 - 54\frac{\sigma}{E}\right]\left(\frac{a}{R}\right)^4\right.$$

$$\left. + \frac{9}{4}\left(\frac{a}{R}\right)^6\right\}$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right)\pi\frac{\sigma^2}{E}\left\{198\left(\frac{\sigma}{E}\right) - \left[\frac{903}{32} - 36\left(\frac{\sigma}{E}\right) + 324\left(\frac{\sigma}{E}\right)^2\right]\left(\frac{a}{R}\right)^2 - \left[3 - 54\left(\frac{\sigma}{E}\right)\right]\left(\frac{a}{R}\right)^4\right.$$

$$\left. - \frac{9}{4}\left(\frac{a}{R}\right)^6 - 12\left(\frac{\sigma}{E}\right)^2\left[42 + \left(\frac{E\frac{t}{R}}{\sigma}\right)^2\right]\frac{1}{\left(\frac{a}{R}\right)^2}\right\}$$

Recalculation of Strain energy increase due to the presence of a hole: 289

$$\hat{\sigma}_{rr} = \frac{1}{2}\sigma\left(1-\frac{a^2}{r^2}\right)\left[1+\left(1-3\frac{a^2}{r^2}\right)\cos 2\theta\right]$$

$$\hat{\sigma}_{\theta\theta} = \frac{1}{2}\sigma\left[\left(1+\frac{a^2}{r^2}\right)-\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\hat{\sigma}_{r\theta} = -\frac{1}{2}\sigma\left(1-\frac{a^2}{r^2}\right)\left(1+3\frac{a^2}{r^2}\right)\sin 2\theta$$

$$\hat{\sigma}_{rr_0} = \frac{1}{2}\sigma\left(1+\cos 2\theta\right)$$

$$\hat{\sigma}_{\theta\theta_0} = \frac{1}{2}\sigma\left(1-\cos 2\theta\right)$$

$$\hat{\sigma}_{r\theta_0} = -\frac{1}{2}\sigma\sin 2\theta$$

Strain energy increase due to the presence of a hole

$$= \frac{\pi a^2 t\sigma^2}{8E}\int_1^\infty \left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left[\left(1-\frac{a^2}{r^2}\right)^2\left\{2+\left(1-3\frac{a^2}{r^2}\right)^2\right\}+2\left(1+\frac{a^2}{r^2}\right)^2+\left(1+\frac{3a^4}{r^4}\right)^2\right.$$
$$\left.+\,2\left(1-\frac{a^2}{r^2}\right)^2\left(1+3\frac{a^2}{r^2}\right)^2-8\right]$$

$$= \frac{\pi a^2 t\sigma^2}{8E}\int_1^\infty \left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left[\left(1-2\frac{a^2}{r^2}+\frac{a^4}{r^4}\right)\left(3-6\frac{a^2}{r^2}+9\frac{a^4}{r^4}\right)+2\left(1+2\frac{a^2}{r^2}+\frac{a^4}{r^4}\right)\right.$$
$$\left.+\left(1+6\frac{a^4}{r^4}+9\frac{a^8}{r^8}\right)+2\left(1+4\frac{a^2}{r^2}-2\frac{a^4}{r^4}-12\frac{a^6}{r^6}+9\frac{a^8}{r^8}\right)-8\right]$$

$$= \frac{\pi a^2 t\sigma^2}{2E}\int_1^\infty \left(\frac{r}{a}\right)d\left(\frac{r}{a}\right)\left[7\left(\frac{a}{r}\right)^4-12\left(\frac{a}{r}\right)^6+9\left(\frac{a}{r}\right)^8\right] = \frac{\pi a^2 t\sigma^2}{2E}\left[\frac{1}{2}+\frac{9}{6}\right]$$

$$\therefore \boxed{\frac{\Delta W_1}{R^3} = \pi \left(\frac{a}{R}\right)^2 \left(\frac{t}{R}\right) \frac{\sigma^2}{E}}$$

Referring to page 81   Impossible !!!

PART (III)

Effect of an Elliptic Hole

Ref: Coker & Filon: Photoelasticity pp 540 - 544

If we write

$$\chi_1 = e^{2\xi} + \cos 2\eta$$

$$\chi_2 = e^{-2\xi} + \cos 2\eta$$

$$\chi_3 = e^{-2\xi} \cos 2\eta$$

$$\chi_4 = \xi$$

$$\chi_5 = e^{2\xi} \cos 2\eta$$

then it is found that the stress function can be written as

$$\chi = \frac{1}{16} T \left\{ \chi_1 + (2e^{2\alpha} - 1)\chi_2 - e^{4\alpha}\chi_3 + 4(1 - \cosh 2\alpha)\chi_4 - \chi_5 \right\}$$

Now the stresses given by the different stress functions are, if we write $(\cosh 2\xi - \cos 2\eta) = 2J^2$,

$$\begin{cases} 2J^4\, \widehat{\xi\xi}_1 = \cos 4\eta - 4 \cos 2\eta \cosh 2\xi + 2 + e^{4\xi} \\ 2J^4\, \widehat{\eta\eta}_1 = \cos 4\eta - 4 \cos 2\eta\, e^{2\xi} + 2 + e^{4\xi} \\ 2J^4\, \widehat{\xi\eta}_1 = 2 \sin 2\eta \cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_2} = \cos 4\eta - 4\cos 2\eta\cosh 2\xi + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\eta\eta_2} = \cos 4\eta - 4\cos 2\eta\; e^{-2\xi} + 2 + e^{-4\xi} \\ 2J^4\,\widehat{\xi\eta_2} = -2\sin 2\eta\cosh 2\xi \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_3} = \cos 4\eta \cdot e^{-2\xi} - \cos 2\eta\,(e^{-4\xi}+3) + 3e^{-2\xi} \\ 2J^4\,\widehat{\eta\eta_3} = -\cos 4\eta \cdot e^{-2\xi} - 3e^{-2\xi} + \cos 2\eta\,(e^{-4\xi}+3) \\ 2J^4\,\widehat{\xi\eta_3} = \sin 4\eta\, e^{-2\xi} - \sin 2\eta\,(e^{-4\xi}+3) \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_4} = \sinh 2\xi \\ 2J^4\,\widehat{\eta\eta_4} = -\sinh 2\xi \\ 2J^4\,\widehat{\xi\eta_4} = \sin 2\eta \end{cases}$$

$$\begin{cases} 2J^4\,\widehat{\xi\xi_5} = \cos 4\eta\, e^{2\xi} - \cos 2\eta\,(e^{4\xi}+3) + 3e^{2\xi} \\ 2J^4\,\widehat{\eta\eta_5} = -\cos 4\eta\, e^{2\xi} + \cos 2\eta\,(e^{4\xi}+3) - 3e^{2\xi} \\ 2J^4\,\widehat{\eta\xi_5} = -\sin 4\eta\, e^{2\xi} + \sin 2\eta\,(e^{4\xi}+3) \end{cases}$$

To find the strain energy increase in the specimen, it is best to find the increase in work done by the external forces, because the difficulty of carrying out the integration in elliptical coördinates.

We have

$$\left.\begin{cases} 2\mu J u_1 = (2-4\sigma)\, e^{2\xi} - (4-4\sigma)\cos 2\eta \\ 2\mu J v_1 = -(2-4\sigma)\sin 2\eta \end{cases}\right\} \text{ due to } X_1$$

$$\left.\begin{cases} 2\mu J u_2 = (4-4\sigma)\cos 2\eta - (2-4\sigma)\, e^{-2\xi} \\ 2\mu J u_2 = -(2-4\sigma)\sin 2\eta \end{cases}\right\} \text{ due to } X_2$$

$$\left.\begin{cases} 2\mu J u_3 = 2\, e^{-2\xi}\cos 2\eta \\ 2\mu J v_3 = 2\, e^{-2\xi}\sin 2\eta \end{cases}\right\} \text{ due to } X_3$$

$$\left\{\, 2\mu J v_4 = -1 \right. \qquad \text{ due to } X_4$$

$$\left.\begin{cases} 2\mu J v_5 = -2\, e^{2\xi}\cos 2\eta \\ 2\mu J v_5 = 2\, e^{2\xi}\sin 2\eta \end{cases}\right\} \text{ due to } X_5$$

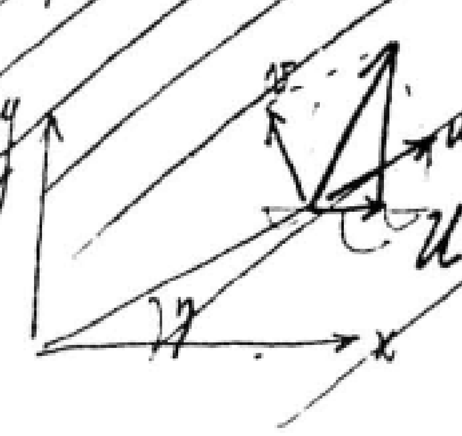
Therefore the total displacement

$$2\mu J u = \frac{1}{16}T\Big[(2-4\sigma)e^{2\xi} - (4-4\sigma)\cos 2\eta + (2e^{2\alpha}-1)\Big\{(4-4\sigma)\cos 2\eta - (2-4\sigma)e^{-2\xi}\Big\}$$
$$- e^{4\alpha}\, 2e^{-2\xi}\cos 2\eta - 4(1-\cosh 2\alpha) + 2e^{2\xi}\cos 2\eta\Big]$$

$$2\mu J v = \frac{1}{16}T\Big[-(2-4\sigma)\sin 2\eta - (2e^{2\alpha}-1)(2-4\sigma)\sin 2\eta - 2e^{4\alpha}e^{-2\xi}\sin 2\eta$$
$$- 2e^{2\xi}\sin 2\eta\Big]$$

$$\begin{cases} 2\mu J u = \dfrac{T}{8}\Big[(1-2\sigma)e^{2\xi} - (2-2\sigma)\cos 2\eta + (2e^{2\alpha}-1)\Big\{(2-2\sigma)\cos 2\eta - (1-2\sigma)e^{-2\xi}\Big\} \\ \qquad - e^{4\alpha}e^{-2\xi}\cos 2\eta - 2(1-\cosh 2\alpha) + e^{2\xi}\cos 2\eta\Big] \\ 2\mu J v = -\dfrac{T}{8}\sin 2\eta\Big[(1-2\sigma) + (2e^{2\alpha}-1)(1-2\sigma) + e^{4\alpha}e^{-2\xi} + e^{2\xi}\Big] \end{cases}$$

~~The component displacement in the direction of tension:~~



$$\cancel{U = u\cos\eta - v\sin\eta}$$

The uniform tension

$$X_0 = \frac{1}{2} T y^2 = \frac{T}{2} \sinh^2 \xi \sin^2 \eta$$

$$= \frac{T}{8} (\cosh 2\xi - 1)(1 - \cos 2\eta)$$

$$= \frac{T}{16} \left\{ (e^{2\xi} + \cos 2\eta) + (e^{-2\xi} + \cos 2\eta) - e^{2\xi} \cos 2\eta - e^{-2\xi} \cos 2\eta - 2 \right\}$$

$$= \frac{T}{16} \left\{ X_1 + X_2 - X_3 - X_5 \right\}$$

Thus the displacements

$$2\mu J u_0 = \frac{T}{8} \left\{ 2(1-2\sigma) \sinh 2\xi + 2 \sinh 2\xi \cos 2\eta \right\}$$

$$2\mu J v_0 = -\frac{T}{8} \left\{ 2(1-2\sigma) \sin 2\eta + 2 \cosh 2\xi \sin 2\eta \right\}$$

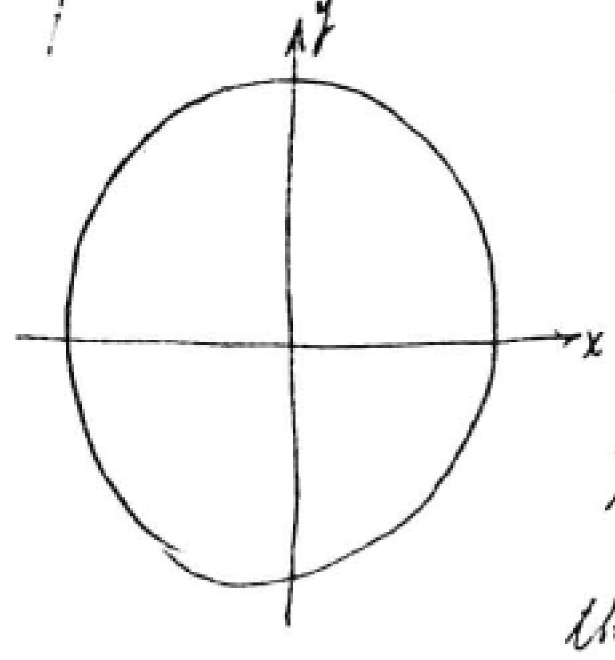
$$= -\frac{T}{8} \sin 2\eta \left\{ 2(1-2\sigma) + 2\cosh 2\xi \right\}$$

~~$\widehat{\xi\xi} = T \cos \eta$ $\quad$ $\widehat{\xi\eta} = -T \sin \eta$ when $\xi = \infty$~~

$$2\mu J(u - u_0) = \frac{T}{8} \left\{ (1-2\sigma) e^{-2\xi} - 2(1-\sigma) \cos 2\eta + (2e^{2\alpha} - 1)\left\{ 2(1-\sigma) \cos 2\eta - (1-2\sigma) e^{-2\xi} \right\} - e^{4\alpha} e^{-2\xi} \cos 2\eta - 2(1 - \cosh 2\alpha) + e^{-2\xi} \cos 2\eta \right\}$$

$$2\mu J(v - v_0) = -\frac{T}{8} \sin 2\eta \left\{ 2(e^{2\alpha} - 1) + e^{4\alpha} e^{-2\xi} - e^{-2\xi} \right\}$$

Now consider the circle at infinity



$$\sqrt{x^2+y^2} = \frac{c}{2} e^{\xi}$$

$$J^2 = \frac{1}{2}\frac{1}{2} e^{2\xi} c^2, \quad \therefore \quad J = \frac{c}{2} e^{\xi}$$

The work done by external forces will to be the shear $\widehat{\xi\eta} + \widehat{\xi\xi}$.

$$\widehat{\xi\eta} = -\frac{1}{2} T \sin 2\eta$$

$$\widehat{\xi\xi} = \frac{1}{2} T (1+\cos 2\eta)$$

Increase in strain energy

$$= \frac{T}{32\mu}\frac{T}{2}\left[\int_0^{2\pi} (1+\cos 2\eta)\left\{4(1-\sigma)(e^{2\alpha}-1)\cos 2\eta - 2(1-\cosh 2\alpha)\right\} d\eta \right.$$

$$\left. + \int_0^{2\pi} \sin^2 2\eta \cdot 2(e^{2\alpha}-1)\, d\eta \right]$$

$$= \frac{T^2}{64\mu}\pi \left[4(1-\sigma)(e^{2\alpha}-1) + 4(\cosh 2\alpha - 1) + 2(e^{2\alpha}-1)\right]$$

$$= \frac{T^2}{32\mu}\pi \left[(3-2\sigma)(e^{2\alpha}-1) + 2(\cosh 2\alpha - 1)\right]$$

$$= \frac{T^2}{16\mu}\pi \left[(3-2\sigma) e^{\alpha} \sinh\alpha + (\cosh 2\alpha - 1)\right] c^2$$

Increase in strain energy $\mathcal{E}$

$$= \frac{(1+\sigma)T^2}{16E}\pi c^2\left[(3-2\sigma)(\sinh\alpha+\cosh\alpha)\sinh\alpha + (\cosh 2\alpha - 1)\right]$$

The axis of the ellipse,

$$a = c\cosh\alpha \qquad a^2-b^2=c^2$$
$$b = c\sinh\alpha$$

$$= t\frac{(1+\sigma)T^2}{16E}\pi\left[(3-2\sigma)(b+a)b + 2b^2\right]$$

$$\mathcal{E} = \frac{(1+\sigma)T^2\pi t}{16E}\left[(5-2\sigma)b^2 + (3-2\sigma)ab\right]$$

$$= \frac{(1+\sigma)T^2}{16E}(\pi ab)t\left[(3-2\sigma) + (5-2\sigma)\left(\frac{b}{a}\right)\right] \qquad \text{O.K.}$$

It is thus shown that the presence of a hole always increases the total strain energy, even compared with the whole flate plate. Therefore we have too much restraining, a buckling stress can only be arrived by considering more accurately the interaction.

Now for the sake of simplicity, go back to the case of a circular buckled region. Here, in order that the buckled circular plate be clamp supported, we choose the form of buckling to be 295

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}^2$$

Thus
$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \\ \frac{1}{R}\frac{\partial w_0}{\partial \theta} = -\frac{1}{2}\left(\frac{r}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial r \partial \theta} = \frac{1}{R}\frac{\partial^2 w_0}{\partial r\partial \theta} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin 2\theta$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{R}\frac{\partial^2 w_0}{\partial \theta^2} = -\left(\frac{r}{R}\right)^2 \cos 2\theta$$

$$\begin{cases} \frac{1}{R}\frac{\partial w}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}\left(\frac{r}{R}\right)^2 \frac{1}{r} \\ \frac{1}{R}\frac{\partial w_0}{\partial r} = -\frac{1}{r}\left(\frac{r}{R}\right)^2 \sin^2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta + 4f\left\{\left(\frac{a}{R}\right)^2 - 3\left(\frac{r}{R}\right)^2\right\}\frac{1}{R^2} \\ \frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta \end{cases}$$

296

$$\frac{1}{r^4}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{r^4}\left(\frac{\partial w_0}{\partial \theta}\right)^2 = 0$$

$$\frac{1}{r^2}\left(\frac{\partial^2 w}{\partial r \partial \theta}\right)^2 - \frac{1}{r^2}\left(\frac{\partial^2 w_0}{\partial r \partial \theta}\right)^2 = 0$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}(\sin^2\theta)^2 - \frac{1}{R^2}\left[\sin^2\theta - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\right]\left[\sin^2\theta - 4f\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$= \frac{1}{R^2}\left[8f\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right)\sin^2\theta - 16f^2\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$= \frac{1}{R^2}\cos 2\theta \cdot 4f\left\{\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right\}$$

$$\nabla^4\phi = \frac{4Ef}{R^2}\left[2\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right)\sin^2\theta + \cos 2\theta\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right) - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

$$= \frac{4Ef}{R^2}\left[\left(\frac{a^2}{R^2} - 2\frac{r^2}{R^2}\right) - \left(\frac{r^2}{R^2}\right)\cos 2\theta - 4f\left(\frac{a^2}{R^2} - \frac{r^2}{R^2}\right)\left(\frac{a^2}{R^2} - 3\frac{r^2}{R^2}\right)\right]$$

If we put $\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{R^2} - f\left\{\frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2}{2}\right\}^2$

$$\nabla^4 \varphi = C\left[p^2-4\right]\left[(p-2)^2-4\right] r^{p-4} \cos 2\theta$$

$$p = 6, \qquad \nabla^4 \varphi = C \cdot 32 \cdot 12\, r^2 \cos 2\theta$$

$$\therefore \qquad C = \frac{-K}{384}$$

$$\therefore \qquad \varphi_4 = -\frac{Ef}{384 R^2}\frac{r^6}{R^2} \cos 2\theta$$

Therefore the particular integral is

$$\boxed{\varphi_0 = \frac{EfR^2}{64}\left[\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\frac{r^4}{R^4} + \frac{2}{9}\left(2f\frac{a^2}{R^2}-1\right)\frac{r^6}{R^6} - \frac{1}{6}\frac{r^6}{R^6}\cos 2\theta - \frac{1}{12} f \frac{r^8}{R^8}\right]}$$

Due to the symmetry of this problem, the solution of the homogeneous equation

$$\nabla^4 \varphi = 0$$

can be written as

$$\frac{\varphi_1}{R^2} = \left(\frac{Ef}{64}\right)\Big[\frac{1}{4} B_0 \frac{r^2}{R^2} + D_0 + \cos 2\theta\left[P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 + \cdots\right]$$

$$+ \cancel{\cos 4\theta\left[P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6\right]} + \cos 6\theta\left[P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8\right]\Big]$$

$$\frac{\varphi}{R^2} = \left(\frac{Ef}{64}\right)\Bigg[\Big\{ + \frac{1}{4}Q_0\left(\frac{r}{R}\right)^2 + \left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^4 + \frac{2}{9}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^6 - \frac{1}{12}f\left(\frac{r}{R}\right)^8\Big\}$$

$$+ \cos 2\theta \left\{ P_2\left(\frac{r}{R}\right)^2 + R_2\left(\frac{r}{R}\right)^4 + \quad - \frac{1}{6}\left(\frac{r}{R}\right)^6 \right\}$$

$$\cancel{+ \cos 4\theta \left\{ P_4\left(\frac{r}{R}\right)^4 + R_4\left(\frac{r}{R}\right)^6 \right\}}$$

$$\cancel{+ \cos 6\theta \left\{ P_6\left(\frac{r}{R}\right)^6 + R_6\left(\frac{r}{R}\right)^8 \right\}}$$

$$\frac{1}{r}\frac{\partial\varphi}{\partial r} = \left(\frac{Ef}{64}\right)\Bigg[\Big\{ \frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^2\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}f\left(\frac{r}{R}\right)^6\Big\}$$

$$+ \cos 2\theta \left\{ 2P_2 + 4R_2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4 \right\}$$

$$\cancel{+ \cos 4\theta \left\{ 4P_4\left(\frac{r}{R}\right)^2 + 6R_4\left(\frac{r}{R}\right)^4 \right\}}$$

$$\cancel{+ \cos 6\theta \left\{ 6P_6\left(\frac{r}{R}\right)^4 + 8R_6\left(\frac{r}{R}\right)^6 \right\}}$$

$$\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{Ef}{64}\Bigg[-4\cos 2\theta \left\{ P_2 + R_2\left(\frac{r}{R}\right)^2 - \frac{1}{6}\left(\frac{r}{R}\right)^4 \right\}$$

$$\cancel{- 16\cos 4\theta \left\{ P_4\left(\frac{r}{R}\right)^2 + R_4\left(\frac{r}{R}\right)^4 \right\}}$$

$$\cancel{- 36\cos 6\theta \left\{ P_6\left(\frac{r}{R}\right)^4 + R_6\left(\frac{r}{R}\right)^6 \right\}} \Bigg]$$

$$\widehat{rr} = \frac{Ef}{64}\Big[\Big\{\tfrac{1}{2}Q_0 + 4\left(\tfrac{a}{R}\right)^2\left(1 - f\tfrac{a^2}{R^2}\right)\left(\tfrac{r}{R}\right)^2 + \tfrac{4}{3}\left(2f\tfrac{a^2}{R^2} - 1\right)\left(\tfrac{r}{R}\right)^4 - \tfrac{2}{3}f\left(\tfrac{r}{R}\right)^6\Big\}$$

$$- \cos 2\theta \Big\{ 2P_2 + \tfrac{1}{3}\left(\tfrac{r}{R}\right)^4 \Big\}$$

$$- \cos 4\theta \Big\{ 12 P_4\left(\tfrac{r}{R}\right)^2 + 10 R_4 \left(\tfrac{r}{R}\right)^4 \Big\}$$

$$- \cos 6\theta \Big\{ 30 P_6 \left(\tfrac{r}{R}\right)^4 + 28 R_6 \left(\tfrac{r}{R}\right)^6 \Big\}\Big]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{\tfrac{1}{2}Q_0 + 12\tfrac{a^2}{R^2}\left(1 - f\tfrac{a^2}{R^2}\right)\left(\tfrac{r}{R}\right)^2 + \tfrac{20}{3}\left(2f\tfrac{a^2}{R^2} - 1\right)\left(\tfrac{r}{R}\right)^4 - \tfrac{14}{3}f\left(\tfrac{r}{R}\right)^6\Big\}$$

$$+ \cos 2\theta \Big\{ P_2 + 12 R_2\left(\tfrac{r}{R}\right)^2 - 5\left(\tfrac{r}{R}\right)^4 \Big\} \qquad !!!$$

$$+ \cos 4\theta \Big\{ 12 P_4 \left(\tfrac{r}{R}\right)^2 + 30 R_4 \left(\tfrac{r}{R}\right)^4 \Big\}$$

$$+ \cos 6\theta \Big\{ 30 P_6 \left(\tfrac{r}{R}\right)^4 + 56 R_6 \left(\tfrac{r}{R}\right)^6 \Big\}$$

$$\widehat{r\theta} = \left(\frac{Ef}{64}\right)\Big[2\sin 2\theta \Big\{ P_2 + 3R_2\left(\tfrac{r}{R}\right)^2 - \tfrac{5}{6}\left(\tfrac{r}{R}\right)^4 \Big\}$$

$$+ 4 \sin 4\theta \Big\{ 3 P_4 \left(\tfrac{r}{R}\right)^2 + 5 R_4 \left(\tfrac{r}{R}\right)^4 \Big\}$$

$$+ 6 \sin 6\theta \Big\{ 5 P_6 \left(\tfrac{r}{R}\right)^4 + 7 R_6 \left(\tfrac{r}{R}\right)^6 \Big\}\Big]$$

300

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left(\widehat{rr} - \nu\,\widehat{\theta\theta}\right)$$

$$\therefore\ \frac{u}{R} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{1}{2}Q_0\left(\frac{r}{R}\right) + \frac{4}{3}(1-3\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^3 + \frac{4}{15}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^5 - \frac{2}{21}(1-7\nu)f\left(\frac{r}{R}\right)^7\Big\}$$

$$-\cos 2\theta\left\{(2+\nu)P_2\left(\frac{r}{R}\right) + 4\nu R_2\left(\frac{r}{R}\right)^3 + \frac{1}{15}(1-15\nu)\left(\frac{r}{R}\right)^5\right\}$$

$$-\cos 4\theta\left\{4(1+\nu)P_4\left(\frac{r}{R}\right)^3 + 2(1+3\nu)R_4\left(\frac{r}{R}\right)^5\right\}$$

$$-\cos 6\theta\left\{6(1+\nu)P_6\left(\frac{r}{R}\right)^5 + 4(1+2\nu)R_6\left(\frac{r}{R}\right)^7\right\} + F(\theta)$$

$$\frac{1}{E}\left(\widehat{\theta\theta} - \nu\,\widehat{rr}\right) = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 4(3-\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{r}{R}\right)^6\Big\}$$

$$+\cos 2\theta\left\{(1+2\nu)P_2 + 12R_2\left(\frac{r}{R}\right)^2 - \left(5-\frac{1}{3}\nu\right)\left(\frac{r}{R}\right)^4\right\}$$

$$+\cos 4\theta\left\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2 + 10(3+\nu)R_4\left(\frac{r}{R}\right)^4\right\}$$

$$+\cos 6\theta\left\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4 + 28(2+\nu)R_6\left(\frac{r}{R}\right)^6\right\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2-\left(\frac{\partial w_0}{\partial r}\right)^2\right\}+\frac{\partial u}{\partial r}=\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)$$

here

$$\frac{1}{E}\left(\widehat{rr}-\nu\widehat{\theta\theta}\right)=\frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2}+4(1-3\nu)\frac{a^2}{R^2}\left(1-f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2$$

$$+\frac{4}{3}(1-5\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{r}{R}\right)^4-\frac{2}{3}(1-7\nu)f\left(\frac{r}{R}\right)^6\Big\}$$

$$-\cos 2\theta\left\{(2+\nu)P_2+12\nu R_2\left(\frac{r}{R}\right)^2-\left(\frac{1}{3}-5\nu\right)\left(\frac{r}{R}\right)^4\right\}$$

$$-\cos 4\theta\left\{12(1+\nu)P_4\left(\frac{r}{R}\right)^2+10(1+3\nu)R_4\left(\frac{r}{R}\right)^4\right\}$$

$$-\cos 6\theta\left\{30(1+\nu)P_6\left(\frac{r}{R}\right)^4+28(1+2\nu)R_6\left(\frac{r}{R}\right)^6\right\}$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2-\left(\frac{\partial w_0}{\partial r}\right)^2\right\}=\frac{1}{2}\left[\left\{\left(\frac{r}{R}\right)\sin^2\theta-4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\frac{r}{R}\right\}^2-\left\{\frac{r}{R}\sin^2\theta\right\}^2\right]$$

$$=\frac{1}{2}\left(\frac{r}{R}\right)^2 4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\left[4f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)-2\sin^2\theta\right] \quad * \; 4f=f$$

$$=\frac{f}{2}\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)\frac{r^2}{R^2}\left[f\left(\frac{a^2}{R^2}-\frac{r^2}{R^2}\right)+\cos 2\theta-1\right]$$

$$=\frac{f}{64}\Big[\Big\{32\frac{a^2}{R^2}\left(f\frac{a^2}{R^2}-1\right)\frac{r^2}{R^2}-32\left(2f\frac{a^2}{R^2}-1\right)\frac{r^4}{R^4}+32f\left(\frac{r}{R}\right)^6\Big\}$$

$$+\cos 2\theta\left\{32\frac{a^2}{R^2}\frac{r^2}{R^2}-32\frac{r^4}{R^4}\right\}\Big]$$

$$\frac{\partial u}{\partial r} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2$$

$$+\frac{20}{3}(5-\nu)\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{14}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^6\Big\}$$

$$-\cos 2\theta\Big\{(2+\nu)P_2 + 4\Big(8\frac{a^2}{R^2}+3\nu R_2\Big)\Big(\frac{r}{R}\Big)^2 - \Big(\frac{97}{3}-5\nu\Big)\Big(\frac{r}{R}\Big)^4\Big\}$$

$$-\cos 4\theta\Big\{12(1+\nu)P_4\Big(\frac{r}{R}\Big)^2 + 10(1+5\nu)R_4\Big(\frac{r}{R}\Big)^4\Big\}$$

$$-\cos 6\theta\Big\{30(1+\nu)P_6\Big(\frac{r}{R}\Big)^4 + 28(1+2\nu)R_6\Big(\frac{r}{R}\Big)^6\Big\}$$

$$\frac{u}{R} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2}\Big(\frac{r}{R}\Big) + 4(3-\nu)\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^3 + \frac{4}{3}(5-\nu)\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^5$$

$$-\frac{2}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^7\Big\}$$

$$-\cos 2\theta\Big\{(2+\nu)P_2\Big(\frac{r}{R}\Big) + \frac{4}{3}\Big(8\frac{a^2}{R^2}+3\nu R_2\Big)\Big(\frac{r}{R}\Big)^3 - \Big(\frac{97}{15}-\nu\Big)\Big(\frac{r}{R}\Big)^5\Big\}$$

$$-\cos 4\theta\Big\{4(1+\nu)P_4\Big(\frac{r}{R}\Big)^3 + 2(1+3\nu)R_4\Big(\frac{r}{R}\Big)^5\Big\}$$

$$-\cos 6\theta\Big\{6(1+\nu)P_6\Big(\frac{r}{R}\Big)^5 + 4(1+9\nu)R_6\Big(\frac{r}{R}\Big)^7\Big\}\Big] + F(\theta)$$

$$\therefore \frac{v}{R} = \frac{p}{64}\Big[\sin 2\theta\Big\{\frac{3}{2}(1+\nu)P_2\left(\frac{r}{R}\right) + 2\left(\frac{8}{3}\frac{a^2}{R^2} + \overline{3+\nu}\,R_2\right)\left(\frac{r}{R}\right)^3 - \left(\frac{86}{15} - \frac{2}{3}\nu\right)\left(\frac{r}{R}\right)^5\Big\}$$

$$+ \sin 4\theta\Big\{4(1+\nu)P_4\left(\frac{r}{R}\right)^3 + 2(4+2\nu)R_4\left(\frac{r}{R}\right)^5\Big\}$$

$$+ \sin 6\theta\Big\{6(1+\nu)P_6\left(\frac{r}{R}\right)^5 + 2(5+3\nu)R_6\left(\frac{r}{R}\right)^7\Big\}\Big] - \int F(\theta)\,d\theta + G\left(\frac{r}{R}\right)$$

$$\frac{\int F(\theta)d\theta}{\left(\frac{r}{R}\right)} + \frac{F'(\theta)}{\left(\frac{r}{R}\right)} + G'\left(\frac{r}{R}\right) - \frac{G\left(\frac{r}{R}\right)}{\left(\frac{r}{R}\right)} = 0$$

or $$\int F(\theta)d\theta + F'(\theta) = G\left(\frac{r}{R}\right) - \frac{r}{R}G'\left(\frac{r}{R}\right)$$

or $$\int F(\theta)d\theta + F'(\theta) = C$$

$$G\left(\frac{r}{R}\right) - \frac{r}{R}G'\left(\frac{r}{R}\right) = C$$

$\therefore F = G = 0$

or $$F(\theta) + F''(\theta) = 0$$

$$F'' + F = 0 \quad \text{only} \quad F = A\,\begin{matrix}\sin\theta\\ \cos\theta\end{matrix} > \text{cnt}$$

$\frac{1}{3} \times \frac{2}{3}\nu$

$$G'\left(\frac{r}{R}\right) - \frac{1}{\left(\frac{r}{R}\right)}G\left(\frac{r}{R}\right) = -\frac{C}{\left(\frac{r}{R}\right)}$$

$$\left(\frac{r}{R}\right)\frac{d}{d\left(\frac{r}{R}\right)}\left[\frac{1}{\left(\frac{r}{R}\right)}G\left(\frac{r}{R}\right)\right] = -\frac{C}{\left(\frac{r}{R}\right)} \quad \Big\| \quad \frac{1}{\left(\frac{r}{R}\right)}G = \frac{C}{\left(\frac{r}{R}\right)} + B$$

$$G = C + B\left(\frac{r}{R}\right)$$

The undisturbed stress function outside the circular region

$$\frac{\phi_1}{R^2} = \frac{1}{4}\sigma(1-\cos 2\theta)\left(\frac{r}{R}\right)^2$$

The other possible solutions are

$$\frac{\phi_2}{R^2} = \sigma\Bigg[P_0 \log\left(\frac{r}{R}\right) \qquad + \cos 2\theta\left\{ Q_2 \frac{1}{\left(\frac{r}{R}\right)^2} + S_2 \right\}$$

$$+ \cos 4\theta \left\{ Q_4 \frac{1}{\left(\frac{r}{R}\right)^4} + S_4 \frac{1}{\left(\frac{r}{R}\right)^2} \right\}$$

$$+ \cos 6\theta \left\{ Q_6 \frac{1}{\left(\frac{r}{R}\right)^6} + S_6 \frac{1}{\left(\frac{r}{R}\right)^4} \right\} \Bigg]$$

$$\frac{1}{\left(\frac{r}{R}\right)} \frac{\partial\left(\frac{\phi_2}{R^2}\right)}{\partial\left(\frac{r}{R}\right)} = \sigma\Bigg[P_0 \frac{1}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \left\{ - \frac{2Q_2}{\left(\frac{r}{R}\right)^4} \right\}$$

$$+ \cos 4\theta \left\{ - \frac{4Q_4}{\left(\frac{r}{R}\right)^6} - \frac{2S_4}{\left(\frac{r}{R}\right)^4} \right\}$$

$$+ \cos 6\theta \left\{ - \frac{6Q_6}{\left(\frac{r}{R}\right)^8} - \frac{4S_6}{\left(\frac{r}{R}\right)^6} \right\} \Bigg]$$

$$\frac{1}{\left(\frac{r}{R}\right)^2} \frac{\partial^2\left(\frac{\phi_2}{R^2}\right)}{\partial\theta^2} = \sigma\Bigg[-4\cos 2\theta \left\{ \frac{Q_2}{\left(\frac{r}{R}\right)^4} + \frac{S_2}{\left(\frac{r}{R}\right)^2} \right\} - 16\cos 4\theta \left\{ \frac{Q_4}{\left(\frac{r}{R}\right)^6} + \frac{S_4}{\left(\frac{r}{R}\right)^4} \right\}$$

$$- 36\cos 6\theta \left\{ \frac{Q_6}{\left(\frac{r}{R}\right)^8} + \frac{S_6}{\left(\frac{r}{R}\right)^6} \right\} \Bigg]$$

$$\widehat{rr}_1 = \sigma\left[\frac{R_0}{\left(\frac{r}{R}\right)^2} - \cos 2\theta \left\{ \frac{6Q_2}{\left(\frac{r}{R}\right)^4} + \frac{4S_2}{\left(\frac{r}{R}\right)^2} \right\} \right.$$

$$\left. - \cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{18\,S_4}{\left(\frac{r}{R}\right)^4} \right\} - \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{40\,S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

$$\widehat{\theta\theta}_2 = \sigma\left[- \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \cdot \frac{6Q_2}{\left(\frac{r}{R}\right)^4} + \cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{6\,S_4}{\left(\frac{r}{R}\right)^4} \right\} \right.$$

$$\left. + \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{20\,S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

$$\widehat{r\theta}_2 = \sigma\left[-2\sin 2\theta \left\{ \frac{3Q_2}{\left(\frac{r}{R}\right)^4} + \frac{S_2}{\left(\frac{r}{R}\right)^2} \right\} - 4\sin 4\theta \left\{ \frac{5Q_4}{\left(\frac{r}{R}\right)^6} + \frac{3S_4}{\left(\frac{r}{R}\right)^4} \right\} \right.$$

$$\left. - 6\sin 6\theta \left\{ \frac{7Q_6}{\left(\frac{r}{R}\right)^8} + \frac{5S_6}{\left(\frac{r}{R}\right)^6} \right\} \right]$$

Therefore $$\widehat{rr} = \sigma\left[\frac{1}{2} + \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \left\{ \frac{1}{2} - \frac{6Q_2}{\left(\frac{r}{R}\right)^4} - \frac{4S_2}{\left(\frac{r}{R}\right)^2} \right\} \right.$$

$$\left. - \cancel{\cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{18\,S_4}{\left(\frac{r}{R}\right)^4} \right\} - \cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{40\,S_6}{\left(\frac{r}{R}\right)^6} \right\}} \right]$$

$$\widehat{\theta\theta} = \sigma\left[\frac{1}{2} - \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta \left\{ \frac{6Q_2}{\left(\frac{r}{R}\right)^4} - \frac{1}{2} \right\} + \cancel{\cos 4\theta \left\{ \frac{20\,Q_4}{\left(\frac{r}{R}\right)^6} + \frac{6\,S_4}{\left(\frac{r}{R}\right)^4} \right\}} \right.$$

$$\left. + \cancel{\cos 6\theta \left\{ \frac{42\,Q_6}{\left(\frac{r}{R}\right)^8} + \frac{20\,S_6}{\left(\frac{r}{R}\right)^6} \right\}} \right]$$

306

$$\widehat{r\theta} = -\sigma\left[\sin 2\theta\left\{\frac{1}{2} + \frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{2S_2}{\left(\frac{a}{R}\right)^2}\right\} + \sin 4\theta\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{12S_4}{\left(\frac{a}{R}\right)^4}\right\}\right.$$

$$\left. + \sin 6\theta\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{30S_6}{\left(\frac{a}{R}\right)^6}\right\}\right]$$

The stress conditions at the boundary of the circular region are then

$$\sigma\left\{\frac{1}{2} + \frac{R_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 4\left(\frac{a}{R}\right)^4\left(1 - f\frac{a^2}{R^2}\right) + \frac{4}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (1)$$

$$\sigma\left\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} + \frac{4S_2}{\left(\frac{a}{R}\right)^2} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{2P_2 + \frac{1}{3}\left(\frac{a}{R}\right)^4\right\} \quad (2)$$

$$\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{18S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 10R_4\left(\frac{a}{R}\right)^4\right\} \quad (3)$$

$$\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{40S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 28R_6\left(\frac{a}{R}\right)^6\right\} \quad (4)$$

$$\sigma\left\{\frac{1}{2} - \frac{R_0}{\left(\frac{a}{R}\right)^2}\right\} = \frac{Ef}{64}\left\{\frac{1}{2}Q_0 + 12\left(\frac{a}{R}\right)^4\left(1 - f\frac{a^2}{R^2}\right) + \frac{20}{3}\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{17}{3}f\left(\frac{a}{R}\right)^6\right\} \quad (5)$$

$$\sigma\left\{\frac{6Q_2}{\left(\frac{a}{R}\right)^4} - \frac{1}{2}\right\} = \frac{Ef}{64}\left\{P_2 + 12R_2\left(\frac{a}{R}\right)^2 - 5\left(\frac{a}{R}\right)^4\right\} \quad (6)$$

$$\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6} + \frac{6S_4}{\left(\frac{a}{R}\right)^4}\right\} = \frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2 + 30R_4\left(\frac{a}{R}\right)^4\right\} \quad (7)$$

$$\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8} + \frac{20S_6}{\left(\frac{a}{R}\right)^6}\right\} = \frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4 + 56R_6\left(\frac{a}{R}\right)^6\right\} \quad (8)$$

$$-\sigma\left\{\frac{1}{2}+\frac{6Q_2}{\left(\frac{a}{R}\right)^4}+\frac{2S_2}{\left(\frac{a}{R}\right)^2}\right\}=\frac{Ef}{64}\left\{2P_2+6R_2\left(\frac{a}{R}\right)^2-\frac{5}{3}\left(\frac{a}{R}\right)^4\right\} \qquad (9)$$

$$-\sigma\left\{\frac{20Q_4}{\left(\frac{a}{R}\right)^6}+\frac{12S_4}{\left(\frac{a}{R}\right)^4}\right\}=\frac{Ef}{64}\left\{12P_4\left(\frac{a}{R}\right)^2+20R_4\left(\frac{a}{R}\right)^4\right\} \qquad (10)$$

$$-\sigma\left\{\frac{42Q_6}{\left(\frac{a}{R}\right)^8}+\frac{30S_6}{\left(\frac{a}{R}\right)^6}\right\}=\frac{Ef}{64}\left\{30P_6\left(\frac{a}{R}\right)^4+42R_6\left(\frac{a}{R}\right)^6\right\} \qquad (11)$$

$$\frac{u}{R}=\frac{\sigma}{E}\Bigg[\frac{1}{2}(1-\nu)\left(\frac{a}{R}\right)-(1+\nu)R_0\frac{1}{\left(\frac{a}{R}\right)}+\cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{a}{R}\right)+2Q_2(1+\nu)\frac{1}{\left(\frac{a}{R}\right)^3}+\frac{4S_2}{\left(\frac{a}{R}\right)}\right\}$$

$$+\cos 4\theta\left\{4(1+\nu)\frac{Q_4}{\left(\frac{a}{R}\right)^5}+2(3+\nu)\frac{S_4}{\left(\frac{a}{R}\right)^3}\right\}+\cos 6\theta\left\{6(1+\nu)\frac{Q_6}{\left(\frac{a}{R}\right)^7}+4(2+\nu)\frac{S_6}{\left(\frac{a}{R}\right)^5}\right\}\Bigg]$$

$$\frac{1}{E}\left(\sigma_\theta-\nu\sigma_r\right)=\frac{\sigma}{E}\Bigg[\frac{1}{2}(1-\nu)-(1+\nu)R_0\frac{1}{\left(\frac{a}{R}\right)^2}+\cos 2\theta\left\{\frac{1}{2}(1+\nu)+\frac{6(1+\nu)Q_2}{\left(\frac{a}{R}\right)^4}+\frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\}$$

$$+\cos 4\theta\left\{\frac{20(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{6(1+3\nu)S_4}{\left(\frac{a}{R}\right)^4}\right\}+\cos 6\theta\left\{\frac{42(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{30(1+2\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}\Bigg]$$

$$\frac{v}{R}=\frac{\sigma}{E}\Bigg[\sin 2\theta\left\{\frac{2(1+\nu)Q_2}{\left(\frac{a}{R}\right)^3}-\frac{1}{2}(1+\nu)\left(\frac{a}{R}\right)\right\}$$

$$+\sin 4\theta\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^5}+\frac{4\nu S_4}{\left(\frac{a}{R}\right)^3}\right\}+\sin 6\theta\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^7}+\frac{(2+6\nu)S_6}{\left(\frac{a}{R}\right)^5}\right\}\Bigg]$$

The boundary conditions at the periphery of the circular region is thus

$$* \quad \sigma\left\{\frac{1}{2}(1-\nu)-(1+\nu)\frac{R_0}{\left(\frac{a}{R}\right)^2}\right\}=\frac{Ef}{64}\left\{(1-\nu)\frac{Q_0}{2}+4(3-\nu)\left(\frac{a}{R}\right)^4\left(1-f\frac{a^2}{R^2}\right)+\frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{a}{R}\right)^6\right\} \quad (12)$$

(5) − ν(1)

$$-\sigma\left\{\frac{1+\nu}{2}+\frac{2Q_2(1+\nu)}{\left(\frac{a}{R}\right)^4}+\frac{4S_2}{\left(\frac{a}{R}\right)^2}\right\}=\frac{Ef}{64}\left\{(2+\nu)P_2+\frac{4}{3}\left(f\frac{a^2}{R^2}+3\nu R_2\right)\left(\frac{a}{R}\right)^2-\left(\frac{97}{15}-\nu\right)\left(\frac{a}{R}\right)^4\right\} \quad (13)$$

$$-\sigma\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{2(3+\nu)S_4}{\left(\frac{a}{R}\right)^4}\right\}=\frac{Ef}{64}\left\{4(1+\nu)P_4\left(\frac{a}{R}\right)^2+2(1+3\nu)R_4\left(\frac{a}{R}\right)^4\right\} \quad (14)$$

$$-\sigma\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{4(2+\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}=\frac{Ef}{64}\left\{6(1+\nu)P_6\left(\frac{a}{R}\right)^4+4(1+2\nu)R_6\left(\frac{a}{R}\right)^6\right\} \quad (15)$$

$$\sigma\left\{\frac{2(1+\nu)Q_2}{\left(\frac{a}{R}\right)^4}-\frac{1}{2}(1+\nu)\right\}=\frac{Ef}{64}\left\{\frac{3}{2}(1+\nu)P_2+2\left(\frac{f}{3}\frac{a^2}{R^2}+\overline{3+\nu}R_2\right)\left(\frac{a}{R}\right)^2-\left(\frac{86}{15}-\frac{2}{3}\nu\right)\left(\frac{a}{R}\right)^4\right\} \quad (16)$$

$$\sigma\left\{\frac{4(1+\nu)Q_4}{\left(\frac{a}{R}\right)^6}+\frac{4\nu S_4}{\left(\frac{a}{R}\right)^4}\right\}=\frac{Ef}{64}\left\{4(1+\nu)P_4\left(\frac{a}{R}\right)^2+2(4+2\nu)R_4\left(\frac{a}{R}\right)^4\right\} \quad (17)$$

$$\sigma\left\{\frac{6(1+\nu)Q_6}{\left(\frac{a}{R}\right)^8}+\frac{2(1+3\nu)S_6}{\left(\frac{a}{R}\right)^6}\right\}=\frac{Ef}{64}\left\{6(1+\nu)P_6\left(\frac{a}{R}\right)^4+2(5+3\nu)R_6\left(\frac{a}{R}\right)^6\right\} \quad (18)$$

Let us investigate the equations (2), (6), (9), (13) & (16)

$$6q_2 + 4s_2 - \frac{1}{2} = \left\{ 3p_2' + \frac{1}{3}h \right\} \qquad h = \left(\frac{a}{R}\right)^4$$

$$6q_2 - \frac{1}{2} = -\left\{ p_2' + 12 r_2' - 5h' \right\}$$

$$-\left(6q_2 + 2s_2 + \frac{1}{2}\right) = \left\{ +2p_2' + 6r_2' - \frac{5}{3}h' \right\}$$

$$-\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right] = \left\{ (2+\nu)p_2' + 4\nu r_2' + \left(\frac{21}{5}+\nu\right)h' \right\}$$

$$\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu)\right] = \left\{ \frac{3}{2}(1+\nu)p_2' + 2(3+\nu)r_2' - \left(\frac{2}{5} - \frac{2}{3}\nu\right)h' \right\}$$

The question is whether this system of equations are consistent. They are not consistent, so we can only satisfy them approximately, by means of method of least square, thus

$$6\left(6q_2 + 4s_2 - \frac{1}{2} - 3p_2 - \frac{1}{3}h\right) + 6\left(6q_2 - \frac{1}{2} - p_2 - 12r_2 + 5h\right)$$

$$+ 6\left(6q_2 + 2s_2 + \frac{1}{2} + 2p_2 + 6r_2 - \frac{5}{3}h\right)$$

$$+ 2(1+\nu)\left[\frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2 + (2+\nu)p_2 + 4\nu r_2 + \left(\frac{21}{5}+\nu\right)h\right]$$

$$+ 2(1+\nu)\left[2(1+\nu)q_2 - \frac{1}{2}(1+\nu) - \frac{3}{2}(1+\nu)p_2 - 2(3+\nu)r_2 + \left(\frac{2}{5} - \frac{2}{3}\nu\right)h\right] = 0.$$

or
$$\boxed{\begin{aligned}&[108 + 4(1+\nu)^2]q_2 + [36 + 8(1+\nu)]s_2 + [(1-\nu^2) - 6]p_2 + [4(1+\nu)(-3+\nu) - 36]r_2 \\ &\quad + \left[-3 + 18h + 2(1+\nu)\left(\frac{23}{5} + \frac{1}{3}\right)h\right] = 0\end{aligned}} \qquad (A)$$

3/10

$$4\left(6q_2+4s_2-\frac{1}{2}-2p_2-\frac{h}{3}\right)+2\left(6q_2+2s_2+\frac{1}{2}+2p_2+6r_2-\frac{5}{3}h\right)$$

$$+4\left[\frac{1+\nu}{2}+2(1+\nu)q_2+4s_2+(2+\nu)p_2+4\nu r_2+\left(\frac{21}{5}+\nu\right)h\right]=0$$

$$\left[36+8(1+\nu)\right]q_2+36s_2+4(1+\nu)p_2+4(3+4\nu)r_2+\left[1+2\nu+\left(\frac{112}{15}+4\nu\right)h\right]=0 \quad (B)$$

$$2\left(2p_2+\frac{h}{3}-6q_2-4s_2+\frac{1}{2}\right)+\left(p_2+12r_2-5h-6q_2+\frac{1}{2}\right)$$

$$+2\left(2p_2+6r_2-\frac{5}{3}h+6q_2+2s_2+\frac{1}{2}\right)+(2+\nu)\left[(2+\nu)p_2+4\nu r_2+\left(\frac{21}{5}+\nu\right)h\right.$$

$$\left.+\frac{1+\nu}{2}+2(1+\nu)q_2+4s_2\right]+\frac{3}{2}(1+\nu)\left[\frac{3}{2}(1+\nu)p_2+2(3+\nu)r_2-\left(\frac{2}{5}-\frac{2}{3}\nu\right)h\right.$$

$$\left.-2(1+\nu)q_2+\frac{1}{2}(1+\nu)\right]=0.$$

$$\left[-6+(1-\nu^2)\right]q_2+4(1+\nu)s_2+\left[9+(2+\nu)^2+\frac{9}{4}(1+\nu)^2\right]p_2+\left[24+4(2+\nu)\nu+3(1+\nu)(3+\nu)\right]r_2$$

$$+\left[\frac{5}{2}+\frac{1}{2}(1+\nu)\left(\frac{7}{2}+\frac{5}{2}\nu\right)+h\left\{\frac{2}{3}-5-\frac{10}{3}+(2+\nu)\left(\frac{21}{5}+\nu\right)-\frac{3}{2}(1+\nu)\left(\frac{2}{5}-\frac{2}{3}\nu\right)\right\}\right]=0 \quad (C)$$

$$6\left(p_2 + 12 r_2 - 5h - 6q_2 + \frac{1}{2}\right) + 3\left(2p_2 + 6r_2 - \frac{5}{3}h + 6q_2 + 2s_2 + \frac{1}{2}\right)$$ 311

$$+ 2\nu\left[(2+\nu)p_2 + 4\nu r_2 + \left(\frac{21}{5}+\nu\right)h + \frac{1+\nu}{2} + 2(1+\nu)q_2 + 4s_2\right]$$

$$+ (3+\nu)\left[\frac{3}{2}(1+\nu)p_2 + 2(3+\nu)r_2 - \left(\frac{2}{5} - \frac{2}{3}\nu\right)h - 2(1+\nu)q_2 + \frac{1}{2}(1+\nu)\right] = 0.$$

$$\boxed{\begin{aligned}&\left[-18 + 4\nu(1+\nu) - 2(1+\nu)(3+\nu)\right]q_2 + \left[6 + 8\nu\right]s_2\\ &+ \left[12 + 2\nu(2+\nu) + \frac{3}{2}(1+\nu)(3+\nu)\right]p_2 + \left[90 + 8\nu^2 + 2(3+\nu)^2\right]r_2\\ &+ \left[\frac{9}{2} + \nu(1+\nu) + \frac{1}{2}(1+\nu)(3+\nu) + h\left\{-35 + 2\nu\left(\frac{21}{5}+\nu\right) - 2(3+\nu)\left(\frac{1}{5} - \frac{1}{3}\nu\right)\right\}\right] = 0\end{aligned}}$$

(D)

The equations (A), (B), (C), (D) determine $\boxed{p_2, q_2, r_2, s_2}$

$$\widehat{rr}+\widehat{\theta\theta} = \sigma\left[1+\cos 2\theta\left\{-\frac{4S_2}{(\frac{r}{R})^2}\right\}\right]$$

$$\int_0^{2\pi} d\theta\left\{(\widehat{rr}+\widehat{\theta\theta})^2 - 2(1+\nu)(\widehat{rr}\cdot\widehat{\theta\theta} - \widehat{r\theta}^2)\right\}$$

$$= \pi\sigma^2\left[2 + \frac{16 S_2^2}{(\frac{r}{R})^4} - 2(1+\nu)\left\{\frac{1}{2} - \frac{2R_0^2}{(\frac{r}{R})^4} - \left(\frac{1}{2} - \frac{6Q_2}{(\frac{r}{R})^4}\right)^2 + \frac{4S_2}{(\frac{r}{R})^2}\left(\frac{1}{2} - \frac{6Q_2}{(\frac{r}{R})^4}\right)\right.\right.$$

$$\left.\left. - \left(\frac{1}{2} + \frac{6Q_2}{(\frac{r}{R})^4} + \frac{2S_2}{(\frac{r}{R})^2}\right)^2\right\}\right]$$

$$= \pi\sigma^2\left[2 + \frac{16 S_2^2}{(\frac{r}{R})^4} - 2(1+\nu)\left\{(-2R_0^2 - 4S_2^2)\frac{1}{(\frac{r}{R})^4} + (-24S_2Q_2 - 24S_2Q_2)\frac{1}{(\frac{r}{R})^6}\right.\right.$$

$$\left.\left. + (-72Q_2^2)\frac{1}{(\frac{r}{R})^8}\right\}\right]$$

$\mathcal{E}_1$ - Strain energy outside the circular region - the same at uniform stress

$$= \frac{t\sigma^2}{2E}\pi\int_a^\infty r\,dr\left[\frac{16S_2^2}{(\frac{r}{R})^4} + 2(1+\nu)\left\{\frac{2(R_0^2+2S_2^2)}{(\frac{r}{R})^4} + \frac{48S_2Q_2}{(\frac{r}{R})^6} + \frac{72Q_2^2}{(\frac{r}{R})^8}\right\}\right]$$

$$= \frac{t\sigma^2}{2E}\pi R^2\left[8\frac{S_2^2}{(\frac{a}{R})^2} + 2(1+\nu)\left\{\frac{(R_0^2+2S_2^2)}{(\frac{a}{R})^2} + 12\frac{S_2Q_2}{(\frac{a}{R})^4} + 12\frac{Q_2^2}{(\frac{a}{R})^6}\right\}\right]$$

For the extensional strain energy in the circular region

$$\widehat{rr}+\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{r}{R}\Big)^6\Big\}$$

$$+\cos 2\theta\Big\{-P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - \frac{16}{3}\Big(\frac{r}{R}\Big)^4\Big\}$$

$$\int_0^{2\pi} d\theta\Big\{(\widehat{rr}+\widehat{\theta\theta})^2 - 2(1+\nu)\big(\widehat{rr}\,\widehat{\theta\theta} - \widehat{r\theta}^2\big)\Big\}$$

$$= \pi\frac{E^2f^2}{64^2}\Big[2\Big\{Q_0 + 16\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + 8\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{16}{3}f\Big(\frac{r}{R}\Big)^6\Big\}^2$$

$$+\Big\{-P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - \frac{16}{3}\Big(\frac{r}{R}\Big)^4\Big\}^2$$

$$-2(1+\nu)\Big\{2\Big[\frac{1}{2}Q_0 + 4\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\frac{r^2}{R^2} + \frac{4}{3}\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{2}{3}f\Big(\frac{r}{R}\Big)^6\Big]$$

$$\times\Big[\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\Big(1-f\frac{a^2}{R^2}\Big)\frac{r^2}{R^2} + \frac{20}{3}\Big(2f\frac{a^2}{R^2}-1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{14}{3}f\Big(\frac{r}{R}\Big)^6\Big]$$

$$-\Big[2P_2 + \frac{1}{3}\Big(\frac{r}{R}\Big)^4\Big]\Big[P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - 5\Big(\frac{r}{R}\Big)^4\Big] - 4\Big[P_2 + 3R_2\Big(\frac{r}{R}\Big)^2 - \frac{5}{6}\Big(\frac{r}{R}\Big)^4\Big]^2\Big\}\Big]$$

Extensional strain energy in the circular region, E_2

$$= \frac{\pi R^2 t E f^2}{2\times 64^2}\Bigg[2\Bigg\{\frac{1}{2}Q_0^2\left(\frac{a}{R}\right)^2 + 8Q_0\left(\frac{a}{R}\right)^6\left(1-f\frac{a^2}{R^2}\right) + \frac{16^2}{6}\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)^2 + \frac{8}{3}Q_0\left(\frac{a}{R}\right)^6\left(2f\frac{a^2}{R^2}-1\right) - \frac{4}{3}Q_0 f\left(\frac{a}{R}\right)^8$$

$$\left(2f\frac{a^2}{R^2}-1\right)\Big) \qquad + \frac{64}{10}\left(2f\frac{a^2}{R^2}-1\right)^2\left(\frac{a}{R}\right)^{10} +$$

$$+ 32\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)\Big) - \frac{64}{9}f\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^{12} + \frac{8\times 1.6}{21}f^2\left(\frac{a}{R}\right)^{14} - \frac{32\times 1.6}{3}f\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^{12}\Bigg\}$$

$$+ \Bigg\{\frac{1}{2}P_2^2\left(\frac{a}{R}\right)^2 - 6R_2P_2\left(\frac{a}{R}\right)^4 + \frac{16}{9}P_2\left(\frac{a}{R}\right)^6 + 24R_2^2\left(\frac{a}{R}\right)^6 - 16R_2\left(\frac{a}{R}\right)^8 + \frac{16\times 1.6}{3}\left(\frac{a}{R}\right)^{10}\Bigg\}$$

$$-4(1+\nu)\Bigg\{\frac{1}{8}Q_0^2\left(\frac{a}{R}\right)^2 + 2Q_0\left(\frac{a}{R}\right)^6\left(1-f\frac{a^2}{R^2}\right) + 8\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)^2 + \frac{2}{3}Q_0\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^6 - \frac{1}{3}Q_0 f\left(\frac{a}{R}\right)^8$$

$$+ \frac{16}{3}\left(\frac{a}{R}\right)^{10}\left(1-f\frac{a^2}{R^2}\right)\left(2f\frac{a^2}{R^2}-1\right) - \frac{8}{3}f\left(1-f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^{12} + \frac{8}{9}\left(2f\frac{a^2}{R^2}-1\right)^2\left(\frac{a}{R}\right)^{10} - \frac{8}{3}f\left(2f\frac{a^2}{R^2}-1\right)\left(\frac{a}{R}\right)^{12}$$

$$+ \frac{2}{9}f^2\left(\frac{a}{R}\right)^{14}\Bigg\}$$

$$+ 2(1+\nu)\Bigg\{3P_2^2\left(\frac{a}{R}\right)^2 + 12P_2R_2\left(\frac{a}{R}\right)^4 - \frac{44}{18}P_2\left(\frac{a}{R}\right)^6 - \frac{3}{4}R_2\left(\frac{a}{R}\right)^8 + \frac{7}{48}\left(\frac{a}{R}\right)^{10} + 9R_2^2\left(\frac{a}{R}\right)^6\Bigg\}$$

$$+ 8(1+\nu)\Bigg\{\frac{1}{2}P_2^2\left(\frac{a}{R}\right)^2 + \frac{3}{2}P_2R_2\left(\frac{a}{R}\right)^4 + \frac{9}{4}R_2^2\left(\frac{a}{R}\right)^6 - \frac{5}{18}P_2\left(\frac{a}{R}\right)^6 - \frac{5}{16}R_2\left(\frac{a}{R}\right)^8 + \frac{25}{360}\left(\frac{a}{R}\right)^{10}\Bigg\}\Bigg]$$

For the circular region

$$k_1 = \frac{\partial^2 w}{\partial r^2} - \frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\,4f\left\{\left(\frac{a}{R}\right)^2 - 3\left(\frac{r}{R}\right)^2\right\}$$

$$k_2 = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}$$

$$= \frac{1}{R}\,4f\left\{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right\}$$

$$\tau = 0$$

$$k_1 k_2 = \frac{1}{R}\,8f\left\{\left(\frac{a}{R}\right)^2 - 2\left(\frac{r}{R}\right)^2\right\}$$

$$k_1 k_2 = \frac{1}{R^2}\,16f^2\left\{\left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 3\left(\frac{r}{R}\right)^4\right\}$$

The bending strain energy in the circular region

$$16f^2\,\frac{1}{24}\,\frac{t^3}{(1-\nu^2)}\,E\cdot 2\pi\int_0^{\left(\frac{a}{R}\right)}\left[4\left\{\left(\frac{a}{R}\right)^2 - 2\left(\frac{r}{R}\right)^2\right\}^2 - 2(1+\nu)\left\{\left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 3\left(\frac{r}{R}\right)^4\right\}\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$= \frac{8}{3}\,\frac{t^3 E\pi}{(1-\nu^2)}\,f^2\int_0^{\left(\frac{a}{R}\right)}\left[2\left(\frac{a}{R}\right)^4 - 8\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 8\left(\frac{r}{R}\right)^4 - (1+\nu)\left\{\left(\frac{a}{R}\right)^4 - 4\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 3\left(\frac{r}{R}\right)^4\right\}\right]\left(\frac{r}{R}\right)d\left(\frac{r}{R}\right)$$

$$= \frac{8}{3}\,\frac{t^3 E\pi f^2}{(1-\nu^2)}\left[1 - 2 + \frac{4}{3} - (1+\nu)\left(\frac{1}{2} - 1 + \frac{1}{2}\right)\right]\left(\frac{a}{R}\right)^6$$

$$= \frac{8}{9}\,\frac{t^3 E\pi f^2}{(1-\nu^2)}\left(\frac{a}{R}\right)^6 = \mathcal{E}_3 \quad \times 16 \quad [\text{due to } \underline{\underline{4f = f}}]\ !!!$$

The decrease in potential of σ [now - case of uniform stress] 316

$$= tR^2\frac{\sigma^2}{E}\int_0^{2\pi} d\theta\left[\frac{1}{2}(1-\nu)R_0 - \frac{1}{2}(1+\nu)R_0 + \cos^2\theta\left(-2(1+\nu)S_2 + 2S_2\right)\right.$$

$$\left. - d\theta\left[-(1+\nu)S_2\right]\sin^2\theta\right]$$

$$= t\frac{\sigma^2}{E}\pi R^2\left[-2\nu R_0 - 2\nu S_2 + (1+\nu)S_2\right]$$

$$= t\frac{\sigma^2}{E}\pi R^2\left[(1-\nu)S_2 - 2\nu R_0\right] = f(\sigma)$$

In order to simplify the calculation: [Note: diff. from p. 309]!!!

Put $\dfrac{R_0}{\left(\frac{a}{R}\right)^2} = r_0$, $\quad f\dfrac{a^2}{R^2} = g$, $\quad \dfrac{Q_0}{\left(\frac{a}{R}\right)^4} = q_0$

$\dfrac{Q_2}{\left(\frac{a}{R}\right)^4} = q_2$, $\quad \dfrac{S_2}{\left(\frac{a}{R}\right)^2} = s_2$, $\quad \dfrac{P_2}{\left(\frac{a}{R}\right)^4} = p_2$,

$\dfrac{R_2}{\left(\frac{a}{R}\right)^2} = r_2$

Important!!! Change eqn. (A), (B), (C), (D)

With this set of notation, !!!

$$\frac{\mathcal{E}_1}{R^3} = \frac{t}{R}\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[8 s_2^2 + 2(1+\nu)\left\{(r_0^2 + 2s_2^2) + 12 q_0 p_2 + 12 p_2^2\right\}\right]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \frac{8}{9} E\pi g^2 \left(\frac{a}{R}\right)^2 \frac{1}{(1-\nu^2)} \quad \frac{1}{16}$$!!! (4f = f)

$$\frac{\mathcal{W}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{E}\pi\left(\frac{a}{R}\right)^2\left\{(1-\nu)s_2 - 2\nu r_0\right\}$$ note!!!

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{1}{8192} E g^2 \left(\frac{a}{R}\right)^6 \Bigg[\Big\{q_0^2 + 16 q_0(1-g) + \frac{256}{3}(1-g)^2 + \frac{16}{3} q_0(2g-1) - \frac{8}{3} q_0 g$$
$$+ 64(1-g)(2g-1) - \frac{128}{9} g(2g-1) + \frac{64}{5}(2g-1)^2$$
$$+ \frac{256}{21} g^2 - \frac{102.4}{3} g(1-g)\Big\}$$
$$+ \left\{\frac{1}{2} p_2^2 - 6 r_2 p_2 + \frac{16}{9} p_2 + 24 r_2^2 - 16 r_2 + \frac{25.6}{3}\right\}$$
$$-(1+\nu)\Big\{\frac{1}{2} q_0^2 + 8 q_0(1-g) + 32(1-g)^2 + \frac{4}{3} q_0(2g-1) - \frac{4}{3} q_0 g$$
$$+ \frac{64}{3}(1-g)(2g-1) - \frac{32}{3} g(1-g) + \frac{32}{9}(2g-1)^2 - \frac{32}{3} g(2g-1) + \frac{8}{9} g^2\Big\}$$
$$+(1+\nu)\left\{6 p_2^2 + 24 p_2 r_2 - \frac{49}{9} p_2 - \frac{3}{2} r_2 + \frac{7}{24} + 18 r_2^2\right\}\Bigg]$$

The equations for determining the values of r_0 + q_0

$$\frac{1}{2} + r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] q \left\{ \frac{1}{2} q_0 + 4(1-q) + \frac{2}{3}(2q-1) - \frac{2}{3} q \right\}$$

$$\frac{1}{2} - r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] q \left\{ \frac{1}{2} q_0 + 12(1-q) + \frac{32}{3}(2q-1) - \frac{14}{3} q \right\}$$

$$1 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] q \left\{ q_0 + 16(1-q) + 8(2q-1) - \frac{16}{3} q \right\}$$

$$q_0 = \frac{1}{q} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 16(1-q) - 8(2q-1) + \frac{16}{3} q$$

If $\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 = \xi$

$$\boxed{q_0 = \frac{1}{q} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 8 + \frac{16}{3} q} \qquad \boxed{q_0 = \frac{1}{q\xi} - 8 + \frac{16}{3} q}$$

$$r_0 = \left[\frac{E}{64\sigma} \frac{a^2}{R^2}\right] q \left\{ \frac{1}{2q} \frac{64\sigma}{E\left(\frac{a}{R}\right)^2} - 4 + \frac{8}{3} q + \frac{8}{3} - 2q \right\} - \frac{1}{2}$$

$$\boxed{r_0 = \left[\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2\right] q \left\{ \frac{2}{3} q - \frac{4}{3} \right\}} \qquad \boxed{r_0 = \xi q \left(\frac{2}{3} q - \frac{4}{3} \right)}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right) \frac{E\left(\frac{a}{R}\right)}{8192} \Big[\left\{ q_0^2 + \frac{4}{3}(4-3g) q_0 + 54.1333 - 49.7778 g + 26.4127 g^2 \right\}$$

$$+ \left\{ \frac{1}{3} p_2^2 - 6 r_2 p_2 + \frac{16}{9} p_2 + 27 r_2^2 - 15 r_2 + \frac{256}{3} \right\}$$

$$- (1+\nu) \left\{ \frac{1}{2} q_0^2 + 4\left(\frac{4}{3} - g\right) q_0 + 14.2222 - 14.2222 g - 6.2222 g^2 \right\}$$

$$+ (1+\nu) \left\{ 6 p_2^2 + 24 p_2 r_2 - \frac{49}{9} p_2 - \frac{3}{2} r_2 + 18 r_2^2 + \frac{7}{24} \right\} \Big]$$

The four equations for the unknowns r_2, s_2, q_2, p_2 are now:

$\nu = 0.3 \qquad 1+\nu = 1.3 \qquad (1+\nu)^2 = 1.69$

$1-\nu^2 = 0.91$

$$114.76\, q_2 + 46.4\, s_2 - 5.09 (\xi g p_2) - 50.04 (\xi g r_2) + [30.22\, \xi g - 3] = 0$$

$$46.4\, q_2 + 36\, s_2 + 5.20 (\xi g p_2) + 16.8 (\xi g r_2) + [13.3333\, \xi g + 1.6] = 0$$

$$-5.09\, q_2 + 5.2\, s_2 + 19.5625 (\xi g p_2) + 39.63 (\xi g r_2) + [2.29333\, \xi g + 5.2625] = 0$$

$$-25.02\, q_2 + 8.4\, s_2 + 19.815 (\xi g p_2) + 122.50 (\xi g r_2) + [-32.96\, \xi g + 7.035] = 0$$

or 1) $q_2 + 0.40432\, s_2 - 0.044353 (\xi g p_2) - 0.43604 (\xi g r_2) + [0.26333\, \xi g - 0.026142] = 0$

2) $q_2 + 0.77586\, s_2 + 0.11207 (\xi g p_2) + 0.36207 (\xi g r_2) + [0.28736\, \xi g + 0.034483] = 0$

3) $-q_2 + 1.02161\, s_2 + 3.84332 (\xi g p_2) + 7.78585 (\xi g r_2) + [0.45056\, \xi g + 1.03389] = 0$

4) $-q_2 + 0.33573\, s_2 + 0.79197 (\xi g p_2) + 4.89608 (\xi g r_2) + [-1.31735\, \xi g + 0.28118] = 0$

390

$$1.42593 S_2 + 3.79297(\xi g p_2) + 7.34981(\xi g r_2) + [0.71389 \xi g + 1.00775] = 0$$
$$1.79747 S_2 + 3.95539(\xi g p_2) + 8.14792(\xi g r_2) + [0.73792 \xi g + 1.06837] = 0$$
$$1.11159 S_2 + 0.90404(\xi g p_2) + 5.25815(\xi g r_2) + [-1.02999 \xi g + 0.31566] = 0$$

$$S_2 + 2.66421(\xi g p_2) + 5.15440(\xi g r_2) + [0.50065 \xi g + 0.70673] = 0$$
$$S_2 + 2.20053(\xi g p_2) + 4.53299(\xi g r_2) + [0.41053 \xi g + 0.59437] = 0$$
$$S_2 + 0.81329(\xi g p_2) + 4.73030(\xi g r_2) + [-0.92659 \xi g + 0.28397] = 0$$

$$1.85092(\xi g p_2) + 0.42410(\xi g r_2) + [1.42724 \xi g + 0.42276] = 0$$
$$0.46368(\xi g p_2) + 0.62141(\xi g r_2) + [0.09012 \xi g + 0.11236] = 0$$

$$\xi g p_2 + 0.22913(\xi g r_2) + [0.77110 \xi g + 0.22841] = 0$$
$$\xi g p_2 + 1.34017(\xi g r_2) + [0.19436 \xi g + 0.24232] = 0$$

Thus $$\xi g r_2 = \frac{0.57674 \xi g - 0.01391}{1.11104}$$

$$\boxed{\xi g r_2 = 0.51910 \xi g - 0.01252}$$

$$(\xi g p_2) + 1.56930(0.51910 \xi g - 0.01252) + (0.96546 \xi g + 0.47073) = 0$$

$$\boxed{\xi g p_2 = -(0.89004 \xi g + 0.22554)}$$

$$3S_2 - 5.67803(0.89004\xi_g + 0.22554) + 14.41769(0.51910\xi_g - 0.01252) + [-0.01541\xi_g + 1.58507] = 0$$

$$3S_2 - (5.05367\xi_g + 1.28062) + (7.48422\xi_g - 0.18051) + (-0.01541\xi_g + 1.58507) = 0$$

$$\boxed{S_2 = -(0.80505\xi_g + 0.04131)}$$

$$2q_2 - 1.18018(0.80505\xi_g + 0.04131) - 0.06772(0.89004\xi_g + 0.22554) - 0.07397(0.51910\xi_g - 0.01252) + (0.55069\xi_g + 0.00834) = 0$$

$$2q_2 - (0.95010\xi_g + 0.04875) - (0.06027\xi_g + 0.01527) - (0.03840\xi_g - 0.000926) + (0.55069\xi_g + 0.00834) = 0$$

$$\boxed{q_2 = 0.24904\xi_g + 0.02738}$$

We have thus

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(0.80505\,\xi g + 0.04131)^2 + 2(1+\nu)\Big\{(\xi g)^2 (g-2)^2 \frac{4}{9} + 2(0.80505\,\xi g + 0.04131)^2$$

$$-12(0.80505\,\xi g + 0.04131)(0.24904\,\xi g + 0.02738) + 12(0.24904\,\xi g + 0.02738)^2\Big\}\Big]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[6.2\,(0.80505\,\xi g + 0.04131)^2 + 1.15556(\xi g)^2(g^2 - 4g + 4)$$

$$- 31.20(0.24904\,\xi g + 0.02738)(0.55691\,\xi g + 0.01393)\Big]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2\left(1.15556\,g^2 - 4.62222\,g + 4.31326\right) - 0.17159(\xi g) - 0.001320\Big]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\left(\frac{a^2}{R}\right)^2 \frac{\sigma^2}{2E}\Big[(\xi g)^2\left(34.5016\,g^2 - 31.2890\,g + 24.5569\right)$$

$$+(3.73333 - 2.8g)(\xi g)\left\{1 - 8\left(1 - \tfrac{2}{3}g\right)(\xi g)\right\} + 5.3(\xi g)(0.89004\,\xi g + 0.22554)$$

$$- 17.95(\xi g)(0.5190\,\xi g - 0.01252) + 8.3\,(0.89004\,\xi g + 0.22554)^2$$

$$- 25.20\,(0.89004\,\xi g + 0.22554)(0.5190\,\xi g - 0.01252) + 47.4(0.5190\,\xi g - 0.01252)^2\Big]$$

$$\boxed{\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\pi\frac{a^2}{R^2}\frac{\sigma^2}{2E}\Big[(\xi g)^2\left(19.5683\,g^2 + 11.0221\,g - 2.2066\right) + \xi g(5.20071 - 2.8g) + 0.50079\Big]}$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE} \left[\frac{4096}{9(1-\nu^2)} \; \frac{1}{\left(\frac{a}{R}\right)^4} (\xi g)^2 \right]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE} \left[\frac{1}{9(1-\nu^2)} \; \frac{g^2}{\left(\frac{\sigma}{E}\right)^2} \right]$$

$$\boxed{\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE} \left[0.122100 \; \frac{g^2}{\left(\frac{\sigma}{E \frac{t}{R}}\right)^2} \right]}$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE} \left[-1.4 (0.80505 \, \xi g + 0.04131) - 1.2 \, \xi g \left(\frac{2}{3} g - \frac{4}{3}\right) \right]$$

$$\boxed{-\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE} \left[(\xi g)(0.8 g - 0.47293) + 0.057834 \right]}$$

Total potential of the system:

$$\left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{gE} \Big[(\xi g)^2 (20.7239 \, g^2 + 6.3999 \, g + 2.1067) + \xi g (5.0291 - 2.8 g) + 0.49947$$
$$+ 0.122100 \frac{g^2}{k^2} + \xi g (0.8 g - 0.47293) + 0.057834 \Big]$$

If σ is a compression, write

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{T^2}{3E}\Big[\xi^2\left(20.7239\,g^4+6.3999\,g^3+2.1067\,g^2\right)+$$

$$-\ \xi\left(4.5562\,g-2.0\,g^2\right)+0.122100\,\frac{g^2}{k^2}+0.55730\Big]$$

Differentiate against g,

$$\xi^2\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)-\xi\left(4.5562-4.0\,g\right)$$

$$+\ 0.244200\,\frac{g}{k^2}=0$$

$$\text{or}\quad k^2=\frac{0.244200\,g}{\xi(4.5562-4.0g)-\xi^2(82.8956\,g^3+19.1999\,g^2+4.2134\,g)}$$

$$\text{or}\quad (4.5562-4g)=2\xi\left(82.8956\,g^3+19.1999\,g^2+4.2134\,g\right)$$

$$\therefore\quad \xi=\frac{1}{2}\,\frac{(4.5562-4g)}{(82.8956\,g^3+19.1999\,g^2+4.2134\,g)}$$

$$k^2=4\,\frac{0.244200\,g^2(82.8956\,g^2+19.1999\,g+4.2134)}{(4.5662-4g)^2}$$

Now if we minus the energy expression with the quantity $\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\cdot 1$, so that the expression truly represent the difference of total potential of the system in two modes, then

$$\left(\frac{t}{R}\right)\pi\left(\frac{a}{R}\right)^2\frac{\sigma^2}{2E}\Big[\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{20.7239 f^4\left(\frac{a}{R}\right)^8 + 6.3999 f^3\left(\frac{a}{R}\right)^6 + 2.1067 f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 f\left(\frac{a}{R}\right)^2 - 2.0 f^2\left(\frac{a}{R}\right)^4\right\} + 0.122100 f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44270\Big]$$

the minimum condition becomes

$$\left(\frac{E}{64\sigma}\right)^2\left(\frac{a}{R}\right)^4\left\{82.8956 f^3\left(\frac{a}{R}\right)^8 + 19.1999 f^2\left(\frac{a}{R}\right)^6 + 4.2134 f\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562\left(\frac{a}{R}\right)^2 - 4f\left(\frac{a}{R}\right)^4\right\} + 0.244200 f\left(\frac{a}{R}\right)^4\frac{1}{K^2} = 0$$

or if $\left(\frac{a}{R}\right) \neq 0$

$$\boxed{\begin{aligned}&\left(\frac{E}{64\sigma}\right)^2\left\{82.8956 f^3\left(\frac{a}{R}\right)^6 + 19.1999 f^2\left(\frac{a}{R}\right)^4 + 4.2134 f\left(\frac{a}{R}\right)^2\right\}\left(\frac{a}{R}\right)^4\\ &-\left(\frac{E}{64\sigma}\right)\left(\frac{a}{R}\right)^2\left\{4.5562 - 4f\left(\frac{a}{R}\right)^2\right\} + f\left(\frac{a}{R}\right)^2\frac{0.244200}{K^2} = 0\end{aligned}}$$

$$\left(\frac{E}{640}\right)^2\left(\frac{a}{R}\right)^4\left\{145.0673 f^4\left(\frac{a}{R}\right)^8 + 38.3994 f^3\left(\frac{a}{R}\right)^6 + 10.5335 f^2\left(\frac{a}{R}\right)^4\right\}$$

$$-\left(\frac{E}{640}\right)\left(\frac{a}{R}\right)^2\left\{13.6686 f\left(\frac{a}{R}\right)^2 - 8.0 f^2\left(\frac{a}{R}\right)^4\right\} + 0.366300 f^2\left(\frac{a}{R}\right)^4\frac{1}{K^2} - 0.44270 = 0$$

Due to very nature of the conditions, it is easier to proceed as follows.

$$\left(\frac{a}{R}\right)^4\left\{82.8956 g^3 + 19.1999 g^2 + 4.2134 g\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{4.5562 - 4g\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.244200\, g}{K^2} = 0$$

or

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{4.5562 - 4g\right\}}{\left\{82.8956 g^2 + 19.1999 g + 4.2134\right\} g}\left(\frac{a}{R}\right)^2 + \frac{0.244200 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{82.8956 g^2 + 19.1999 g + 4.2134\right\}} = 0}$$

$$\left(\frac{a}{R}\right)^4\left\{145.0673 g^4 + 38.3994 g^3 + 10.5335 g^2\right\}\left(\frac{1}{64K\left(\frac{t}{R}\right)}\right)^2$$

$$-\left(\frac{a}{R}\right)^2\left\{13.6686 g - 8g^2\right\}\frac{1}{64K\left(\frac{t}{R}\right)} + \frac{0.366300 g^2}{K^2} - 0.44270 = 0$$

$$\boxed{\begin{aligned}\left(\frac{a}{R}\right)^4 - \frac{64K\left(\frac{t}{R}\right)\left\{13.6686 - 8g\right\}\left(\frac{a}{R}\right)^2}{\left\{145.0673 g^2 + 38.3994 g + 10.5335\right\} g} + \frac{0.36630 \times 64^2 \times \left(\frac{t}{R}\right)^2}{\left\{145.0673 g^2 + 38.3994 g + 10.5335\right\}} \\ - \frac{0.44270 \times 64^2 \times K^2\left(\frac{t}{R}\right)^2}{\left\{145.0673 g^2 + 38.3994 g + 10.5335\right\} g^2} = 0\end{aligned}}$$

Both equations can be written as

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\{1.13905 - g\}}{\underbrace{\{0.32381 g^2 + 0.075000 g + 0.016459\}}_{X} g}\left(\frac{a}{R}\right)^2 + \frac{3.9072\left(\frac{t}{R}\right)^2}{\{0.32381 g^2 + 0.075000 g + 0.016459\}} = 0}$$

$$\boxed{\left(\frac{a}{R}\right)^4 - \frac{K\left(\frac{t}{R}\right)\{1.70858 - g\}}{\underbrace{\{0.28333 g^2 + 0.075000 g + 0.020573\}}_{W} g}\left(\frac{a}{R}\right)^2 + \frac{(2.9304 g^2 - 3.5416 K^2)\left(\frac{t}{R}\right)^2}{\{0.28333 g^2 + 0.075000 g + 0.020573\} g^2} = 0}$$

$$\left(\frac{a}{R}\right)^2 = \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\{1.13905 - g\}}{X g} \pm \sqrt{\frac{1}{4}\,\frac{K^2\left(\frac{t}{R}\right)^2\{1.13905 - g\}^2}{X^2 g^2} - \frac{3.9072\left(\frac{t}{R}\right)^2}{X}}$$

$$= \frac{1}{2}\,\frac{K\left(\frac{t}{R}\right)\{1.13905 - g\}}{X g}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072\, X g^2}{K^2\{1.13905 - g\}^2}}\right]$$

$$\frac{K^2\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{(1.70858 - g)}{W} - \frac{(1.13905 - g)}{X}\right]\frac{1}{2}\,\frac{(1.13905 - g)}{X}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072\, X g^2}{K^2(1.13905 - g)^2}}\right]$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{g^2}\left[\frac{3.9072 g^2}{X} - \frac{(2.9304 g^2 - 3.5416 K^2)}{W}\right] = 0$$

338

$$\frac{1}{2}\frac{(1.13905-g)}{X}\left[\frac{(1.70858-g)}{W}-\frac{(1.13905-g)}{X}\right]\left[1\pm\sqrt{1-\frac{15.6288\,X g^2}{K^2(1.13905-g)^2}}\right]K^2$$

$$+\left[\frac{3.90729g^2}{X}-\frac{(2.9304g^2-3.5416K^2)}{W}\right]=0$$

where $X = 0.32381g^2 + 0.075000g + 0.016459$

$W = 0.28333g^2 + 0.075000g + 0.020573$

When $g = 0.1$

$X = 0.027197, \quad W = 0.030906$ $\quad \frac{1}{X} = 36.769$

$\frac{1}{W} = 32.356$

$$\frac{1}{2}\frac{1.03905}{0.027197}\left[\frac{1.60858}{0.030906}-\frac{1.03905}{0.027197}\right]\left[1\pm\sqrt{1-\frac{0.156288\times0.027197}{K^2\times1.03905^2}}\right]K^2$$

$$=\frac{0.029304-3.5416K^2}{0.030906}-\frac{0.03.9072}{0.027197}$$

$$1-\frac{0.156288\times0.027197}{K^2\times1.03905^2}=\left[\frac{32.356}{19.1024\times13.8424}\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{36.769\times 3.9072}{19.1024\times13.8424}\frac{1}{K^2}-1\right]^2$$

$$1-\frac{0.0039371}{K^2}=\left[0.12237\left(\frac{0.029304}{K^2}-3.5416\right)-\frac{0.0054331}{K^2}-1\right]^2$$

$$=\left[\frac{0.0018432}{K^2}+0.56161\right]^2$$

$$1-\frac{0.0039371}{K^2} = \left(\frac{0.0018472}{K^2}\right)^2 + \frac{0.0020933}{K^2} + 0.32105$$

$$\left(\frac{0.001}{K^2}\right)^2 3.41215 + \left(\frac{0.001}{K^2}\right) 6.0304 - 0.67895 = 0$$

$$\left(\frac{0.001}{K^2}\right)^2 + 1.7673\left(\frac{0.001}{K^2}\right) - 0.19898 = 0$$

$$\left(\frac{0.001}{K^2}\right) = -0.88365 + \sqrt{0.88365^2 + 0.19898}$$

$$= -0.88365 + \sqrt{0.97982} = -0.88365 + 0.98986$$

$$= 0.10621$$

$$\therefore \quad K^2 = 0.0094153 \qquad \boxed{k = 0.097032}$$

$$\frac{\left(\frac{a}{R}\right)^2}{\left(\frac{t}{R}\right)} = \frac{1}{2}\,\frac{0.97032 \times 1.03905}{0.027197}\left[1 \pm \sqrt{1 - \frac{4 \times 3.9072 \times 0.027197 \times 1.0621}{1.03905^2}}\right]$$

$$= 18.3845 \times 0.97032 \times 1.03905\left[1 \mp \sqrt{0.58183}\right]$$

0.23722
1.76273

$$= 4.397$$

32.674

$$\text{If} \quad \left(\frac{t}{R}\right) = \frac{1}{1000} \qquad \left(\frac{a}{R}\right)^2 = 0.004397 \qquad \boxed{\frac{a}{R} = 0.01135}$$

$$\cancel{f\left(\frac{a}{R}\right)^2 = 0.1}, \qquad \boxed{\cancel{f = \frac{0.1}{0.032674} = 3.060}}$$

$$\frac{\Delta v_{max}}{t} = f\,\frac{\left(\frac{a}{R}\right)^4}{4}\,\frac{R}{t} = \frac{f\left(\frac{a}{R}\right)^2/(t/R)}{4} = \frac{0.1 \times \overset{32.674}{4.397}}{4} = \underline{\underline{\overset{0.8169}{0.1099}}}$$

If we consider (ξg) also as a variable,

$$6q_2 + 4s_2 - \frac{1}{2} = 2\,\xi g\, p_2 + \frac{1}{3}\,\xi g$$

$$6q_2 - \frac{1}{2} = \xi g\, p_2 + 12\,\xi g\, r_2 - 5\,\xi g$$

$$-6q_2 - 2s_2 - \frac{1}{2} = 2\,\xi g\, p_2 + 6\,\xi g\, r_2 - \frac{5}{3}\,\xi g$$

$$-0.65 - 2.6q_2 - 4s_2 = 2.3\,\xi g\, p_2 + 1.2\,\xi g\, r_2 + 4.5\,\xi g$$

$$2.6q_2 - 0.65 = 1.95\,\xi g\, p_2 + 6.6\,\xi g\, r_2 - 0.2\,\xi g$$

This is a system of equations for 5 unknowns, $q_2, s_2, p_2, r_2, (\xi g)$

$$\begin{cases} q_2 + 0.666667\,s_2 - 0.33333\,\xi g\, p_2 + 0 = 0.055556\,\xi g + 0.083333 \\ q_2 + 0 - 0.166667\,\xi g\, p_2 - 2\,\xi g\, r_2 = -0.833333\,\xi g + 0.083333 \\ q_2 + 0.33333\,s_2 + 0.33333\,\xi g\, p_2 + \xi g\, r_2 = +0.277778\,\xi g - 0.083333 \\ q_2 + 1.53846\,s_2 + 0.88461\,\xi g\, p_2 + 0.46154\,\xi g\, r_2 = -1.73077\,\xi g - 0.25000 \\ q_2 + 0 - 0.75000\,\xi g\, p_2 - 2.53846\,\xi g\, r_2 = -0.076923\,\xi g + 0.25000 \end{cases}$$

$$0.666667\,s_2 - 0.166667\,\xi g\, p_2 + 2\,\xi g\, r_2 = 0.888889\,\xi g$$

$$0.33333\,s_2 + 0.5000\,\xi g\, p_2 + 3\,\xi g\, r_2 = 1.111111\,\xi g - 0.166667$$

$$1.20513\,s_2 + 0.55128\,\xi g\, p_2 - 0.53846\,\xi g\, r_2 = -2.00855\,\xi g - 0.16667$$

$$1.53846\,s_2 + 1.63461\,\xi g\, p_2 + 3.0000\,\xi g\, r_2 = -1.65385\,\xi g - 0.50000$$

$$S_2 - 0.25000\,\xi_g p_2 + 3.0000\,\xi_g r_2 = 1.33333\,\xi_g$$
$$S_2 + 1.5000\,\xi_g p_2 + 9.000\,\xi_g r_2 = 3.33333\,\xi_g - 0.50000$$
$$S_2 + 0.45744\,\xi_g p_2 - 0.44681\,\xi_g r_2 = -1.66667\,\xi_g - 0.138298$$
$$S_2 + 1.06250\,\xi_g p_2 + 1.950\,\xi_g r_2 = -1.07500\,\xi_g - 0.32500$$

$$1.75000\,\xi_g p_2 + 6.0000\,\xi_g r_2 = 2.00000\,\xi_g - 0.50000$$
$$1.04256\,\xi_g p_2 + 9.44681\,\xi_g r_2 = 5.0000\,\xi_g - 0.36170$$
$$0.60506\,\xi_g p_2 + 2.39681\,\xi_g r_2 = 0.59167\,\xi_g - 0.18170$$

$$\xi_g p_2 + 3.42857\,\xi_g r_2 = 1.14286\,\xi_g - 0.28571$$
$$\xi_g p_2 + 9.06116\,\xi_g r_2 = 4.79589\,\xi_g - 0.34693$$
$$\xi_g p_2 + 3.96128\,\xi_g r_2 = 0.97787\,\xi_g - 0.30856$$

$$5.63259\,\xi_g r_2 = 3.65303\,\xi_g - 0.06122$$
$$5.09988\,\xi_g r_2 = 3.81802\,\xi_g - 0.03837$$

$$4.21685\,\xi_g - 0.04238 = 3.65303\,\xi_g - 0.06122$$

$$\xi_g = -\frac{0.01884}{0.56382} = -0.03341$$

If compression is taken as positive

$\cancel{\boxed{\xi_g = 0.03341}}$ $\boxed{\xi_g = 0.03341}$

$$r_2 = \frac{-7.47105 \times 0.03341 - 0.09959}{-10.73247 \times 0.03341}$$

$$= \frac{0.09959 + 0.24961}{0.35857} = \boxed{0.97387 = r_2}$$

$$p_2 = \frac{-0.03341(-16.02115 + 6.91662) - 0.94120}{-3 \times 0.03341}$$

$$= -3.03484 + 9.39039 = \boxed{6.35555 = p_2}$$

$$s_2 = \frac{-0.03341(-17.60449 - 13.15035 + 1.92500) - 0.96330}{4}$$

$$= 0$$

$$q_2 = \frac{-0.03341(0.20376 + 2.99652 - 2.30768) + 0.083333}{5}$$

$$= \boxed{0.01070 = q_2}$$

Check $\quad 0.01070 + 0.33333 \times 0.03341 \times 6.3555$

$= 0.08148 \qquad$ checks

$$q_0 = \frac{16}{3}g - 8 + \frac{1}{g\xi} = \frac{16}{3}g - 37.931$$

$$r_0 = \xi g\left(\frac{2}{3}g - \frac{4}{3}\right)$$

$$q_2 = 0.01070$$

$$s_2 = 0$$

$$p_2 = 6.3555$$

$$r_2 = 0.97387$$

$$\xi g = -0.03341$$

$$\xi g = -0.03341$$

$$= \frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g$$

$$\left(\frac{a}{R}\right)^2 = -\frac{64 \times 0.03341}{g}\left(\frac{\sigma}{E}\right)$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{gE}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{\frac{4}{9}(\xi g)^2\left(g^2 - 4g + 4\right) + 12 \times 0.01070^2\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{gE}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\left(g^2 - 4g + 4\right) + 0.0013379\right\}\right]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{gE}\pi\left(\frac{a}{R}\right)^2\left[2.6\left\{0.0004961\,g^2 - 0.0019844\,g + 0.0033223\right\}\right]$$

$$\boxed{\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{gE}\pi\left(\frac{a}{R}\right)^2\left[0.0012899\,g^2 - 0.0051594\,g + 0.0086380\right]}$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(0.03341)^2\Big[0.35(5.3333g - 37.931)^2 + (3.7333 - 2.8000g)$$
$$\times(5.3333g - 37.931)$$

$$+ 15.6444 - 31.2889g + 34.5016g^2 + 8.9125$$

$$+ 8.3 \times 6.3555^2 + 25.2 \times 6.3555 \times 0.97367 - 5.3 \times 6.3555 + 22.05 \times 0.97367$$
$$+ 7.4 \times 0.97367^2\Big]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(0.03341)^2\Big[$$

g^2	g	
$+9.95555$	-141.609	$+503.566$
-14.93333	$+106.207$	-141.609
$+34.5016$	$+19.911$	$+15.644$
29.1238	-31.289	$+8.913$
	-46.780	$+335.257$
		$+155.974$
		-33.684
		$+7.018$
		$+21.474$
		872.553

$$\boxed{\begin{aligned}\frac{\mathcal{E}_2}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[0.032508g^2 - 0.052216g + 0.97394\right]\\ \frac{\mathcal{E}_3}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[0.12210\frac{g^2}{\left(\frac{\sigma}{E}\frac{R}{t}\right)^2}\right]\\ \frac{\mathcal{E}_2}{R^3} &= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{c}\right)^2\left[+0.013367g - 0.026733\right]\times 2 \quad ?\end{aligned}}$$

$$0.033798\,g^2 - 0.084109\,g + 0.03604 + \frac{0.12210\,g^2}{K^2} = 0$$

$$g^2 - 2.4886\,g + 1.0663 = 0$$

$$g = 1.2443 \pm \sqrt{1.2443^2 - 1.0663} = 1.2443 \pm \sqrt{0.4820} = 1.2443 \pm 0.6943$$

$$= 0.5500$$

$$1.9386$$

$$\boxed{K^2 = \frac{g^2}{0.68885\,g - (0.29517 + 0.27661\,g^2)}}$$

①	②	③	④	⑤	⑥	⑦	⑧
g	g^2	$0.68885g$	$0.27661g^2$	③ - (0.29517 + ④)	② / ⑤ = K^2		
0.60	0.36	0.41331	0.09965	0.01849	19.47		
0.70	0.49	0.48220	0.13564	0.05139	9.53		
0.80	0.64	0.55108	0.17716	0.07875	8.13		
0.90	0.81	0.61997	0.22422	0.10058	8.05		
1.00	1.00	0.68885	0.27661	0.11687	8.55		
1.10	1.21	0.75774	0.33494	0.12763	9.49		
1.20	1.44	0.82662	0.39861	0.13284	10.84		
1.40	1.69	0.96439	0.46781	0.20141			
1.60	2.56	1.10216	0.70863	0.09836			
1.80	3.24	1.23993	0.89661	0.04790			

Comparison of different energies

For $g = 0.90$,

(Difference!)

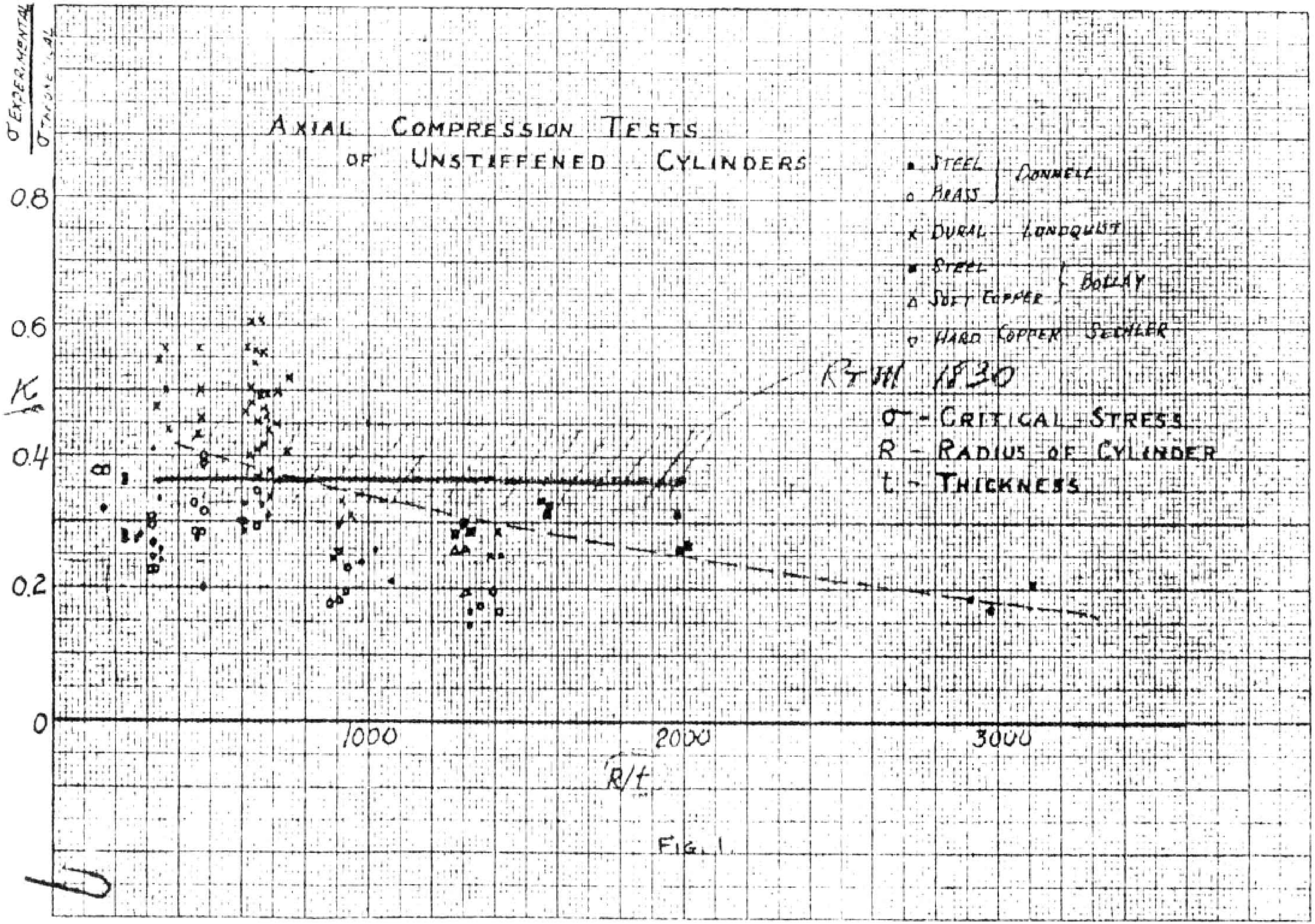
$\frac{\mathcal{E}_1}{R^3}$ = extensional strain energy outside the circular region

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\begin{array}{r} 0.001044 \\ -0.004640 \\ 0.008638 \\ \hline 0.005042 \end{array}\right]$$

$\xi g = -0.03341$

$\frac{E}{64\sigma}\left(\frac{a}{R}\right)^2 g = -0.03341$

$\left(\frac{a}{R}\right)^2 = \frac{-64 \times 0.03341}{g}\frac{\sigma}{E}$



AXIAL COMPRESSION TESTS
OF UNSTIFFENED CYLINDERS
σ EXPERIMENTAL
σ THEORETICAL
0.8
0.6
K
0.4
0.2
0
1000
2000
3000
R/t
• STEEL
○ BRASS
DONNELL
× DURAL LUNDQUIST
▪ STEEL
△ SOFT COPPER
BOLLAY
▫ HARD COPPER SECHLER
σ - CRITICAL STRESS
R - RADIUS OF CYLINDER
t - THICKNESS

FIG. 1

For the circular region:

$$\widehat{rr} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 4\Big(\frac{a}{R}\Big)^2\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + \frac{4}{3}\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{2}{3}f\Big(\frac{r}{R}\Big)^6\Big\}$$
$$- \cos 2\theta\Big\{2P_2 + \frac{1}{3}\Big(\frac{r}{R}\Big)^4\Big\}\Big]$$

$$\widehat{\theta\theta} = \frac{Ef}{64}\Big[\Big\{\frac{1}{2}Q_0 + 12\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + \frac{20}{3}\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4 - \frac{14}{3}f\Big(\frac{r}{R}\Big)^6\Big\}$$
$$+ \cos 2\theta\Big\{2P_2 + 12R_2\Big(\frac{r}{R}\Big)^2 - 5\Big(\frac{r}{R}\Big)^4\Big\}\Big]$$

$$\widehat{r\theta} = \frac{Ef}{64}\Big[\sin 2\theta\Big\{2P_2 + 6R_2\Big(\frac{r}{R}\Big)^2 - \frac{5}{3}\Big(\frac{r}{R}\Big)^4\Big\}\Big]$$

$$\frac{1}{E}(\widehat{rr} - \nu\widehat{\theta\theta}) = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 4(1-3\nu)\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + \frac{4}{3}(1-5\nu)\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4$$
$$- \frac{2}{3}(1-7\nu)f\Big(\frac{r}{R}\Big)^6\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2 + 12\nu R_2\Big(\frac{r}{R}\Big)^2 + \Big(\frac{1}{3} - 5\nu\Big)\Big(\frac{r}{R}\Big)^4\Big\}\Big]$$

$$\frac{1}{2}\Big\{\Big(\frac{\partial w}{\partial r}\Big)^2 - \Big(\frac{\partial w_0}{\partial r}\Big)^2\Big\} = \frac{f}{64}\Big\{32\Big(\frac{a}{R}\Big)^2\Big(f\frac{a^2}{R^2} - 1\Big)\frac{r^2}{R^2} - 32\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4 + 32f\Big(\frac{r}{R}\Big)^6$$
$$+ \cos 2\theta\Big(32\Big(\frac{a}{R}\Big)^2 - 32\Big(\frac{r}{R}\Big)^2\Big)\Big(\frac{r}{R}\Big)^2\Big\}$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{f}{64}\Big[\Big\{(1-\nu)\frac{Q_0}{2} + 12(3-\nu)\frac{a^2}{R^2}\Big(1 - f\frac{a^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^2 + \frac{20}{3}(5-\nu)\Big(2f\frac{a^2}{R^2} - 1\Big)\Big(\frac{r}{R}\Big)^4$$
$$- \frac{14}{3}(7-\nu)f\Big(\frac{r}{R}\Big)^6\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2 + 4\Big(8\frac{a^2}{R^2} + 3\nu R_2\Big)\Big(\frac{r}{R}\Big)^2$$
$$- 5\Big(\frac{19}{3} + \nu\Big)\Big(\frac{r}{R}\Big)^4\Big\}\Big]$$

$$\frac{u}{R} = \frac{f}{64}\Big[\Big\{\frac{(1-\nu)}{2}Q_0\left(\frac{r}{R}\right) + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^3 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^5 - \frac{2}{3}(7-\nu)f\left(\frac{r}{R}\right)^7\Big\} - \cos 2\theta\Big\{2(1+\nu)P_2\left(\frac{r}{R}\right) + \frac{4}{3}\left(8\frac{a^2}{R^2} + 3\nu R_2\right)\left(\frac{r}{R}\right)^3 - \left(\frac{19}{3}+\nu\right)\left(\frac{r}{R}\right)^5\Big\}\Big]$$

$$\frac{1}{E}(\widehat{\theta\theta} - \nu\,\widehat{rr}) = \frac{f}{64}\Big[\Big\{\frac{(1-\nu)}{2}Q_0 + 4(3-\nu)\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{r}{R}\right)^2 + \frac{4}{3}(5-\nu)\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{r}{R}\right)^4 - \frac{2}{3}(7-\nu)f\left(\frac{r}{R}\right)^6\Big\} + \cos 2\theta\Big\{2(1+\nu)P_2 + 12\;R_2\left(\frac{r}{R}\right)^2 - \left(5 - \frac{1}{3}\nu\right)\left(\frac{r}{R}\right)^4\Big\}\Big]$$

$$\frac{1}{r}\frac{\partial v}{\partial\theta} = \frac{f}{64}\cos 2\theta\Big\{4(1+\nu)P_2 + \frac{32}{3}\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^2 + 12(1+\nu)R_2\left(\frac{r}{R}\right)^2 - \frac{2}{3}(17+\nu)\left(\frac{r}{R}\right)^4\Big\}$$

$$\boxed{\frac{v}{R} = \frac{f}{64}\sin 2\theta\Big\{2(1+\nu)P_2\left(\frac{r}{R}\right) + \frac{16}{3}\left(\frac{a}{R}\right)^2\left(\frac{r}{R}\right)^3 + 6(1+\nu)R_2\left(\frac{r}{R}\right)^3 - \frac{1}{3}(17+\nu)\left(\frac{r}{R}\right)^5\Big\}}$$

For the region outside the circle:

$$\widehat{rr} = \sigma\left[\frac{1}{2} + \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta\left\{\frac{1}{2} - \frac{6Q_2}{\left(\frac{r}{R}\right)^4} - \frac{4S_2}{\left(\frac{r}{R}\right)^2}\right\}\right]$$

$$\widehat{\theta\theta} = \sigma\left[\frac{1}{2} - \frac{R_0}{\left(\frac{r}{R}\right)^2} + \cos 2\theta\left\{\frac{6Q_2}{\left(\frac{r}{R}\right)^4} - \frac{1}{2}\right\}\right]$$

$$\widehat{r\theta} = -\sigma\sin 2\theta\left\{\frac{1}{2} + \frac{6Q_2}{\left(\frac{r}{R}\right)^4} + \frac{2S_2}{\left(\frac{r}{R}\right)^2}\right\}$$

343

$$\frac{u}{R} = \frac{\sigma}{E}\left[\frac{1}{2}(1-\nu)\left(\frac{r}{R}\right) - (1+\nu)\frac{R_0}{\left(\frac{r}{R}\right)} + \cos 2\theta\left\{\frac{1}{2}(1+\nu)\left(\frac{r}{R}\right) + 2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} + \frac{4S_2}{\left(\frac{r}{R}\right)}\right\}\right]$$

$$\frac{v}{R} = \frac{\sigma}{E}\sin 2\theta\left\{2(1+\nu)\frac{Q_2}{\left(\frac{r}{R}\right)^3} - \frac{1}{2}(1+\nu)\left(\frac{r}{R}\right)\right\}$$

With the simplified relation of p. 316, the condition of stress continuity becomes

$$\frac{1}{2} + r_0 = \xi g\left\{\frac{1}{2}q_0 + 4(1-g) + \frac{4}{3}(2g-1) - \frac{2}{3}g\right\}$$

$$-\frac{1}{2} + 6q_2 + 4S_2 = \xi g\left\{2p_2 + \frac{1}{3}\right\}$$

$$\frac{1}{2} - r_0 = \xi g\left\{\frac{1}{2}q_0 + 12(1-g) + \frac{20}{3}(2g-1) - \frac{14}{3}g\right\}$$

$$6q_2 - \frac{1}{2} = \xi g\left\{2p_2 + 12r_2 - 5\right\}$$

$$-\frac{1}{2} - 6q_2 - 2S_2 = \xi g\left\{2p_2 + 6r_2 - \frac{5}{3}\right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4S_2 = \xi g\left\{-2(1+\nu)p_2 - \frac{4}{3}(8+30r_2) + \left(\frac{19}{3}+\nu\right)\right\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \xi g\left\{2(1+\nu)p_2 + 6(1+\nu)r_2 - \frac{5}{3}(1+\nu)\right\}$$

Thus

$$\boxed{\begin{aligned} q_0 &= \frac{1}{g\xi} - 8\left(1 - \frac{2}{3}g\right) \\ r_0 &= \xi g\,\frac{2}{3}(g-2). \end{aligned}}$$

$$q_2 + 0.666667\, s_2 - 0.083333 = \xi g \left\{ 0.33333\, p_2 + 0.055556 \right\}$$

$$q_2 + 0 - 0.083333 = \xi g \left\{ 0.33333\, p_2 + 2 r_2 - 0.833333 \right\}$$

$$-q_2 - 0.33333\, s_2 - 0.083333 = \xi g \left\{ 0.33333\, p_2 + r_2 - 0.277778 \right\}$$

$$q_2 + 1.53846\, s_2 + 0.25000 = \xi g \left\{ - p_2 - 0.461538\, r_2 - 1.55128 \right\}$$

$$q_2 - 0.25000 = \xi g \left\{ p_2 + 3 r_2 - 0.155128 \right\}$$

$$0.666667\, s_2 + 0 = \xi g \left\{ -2 r_2 + 0.888889 \right\}$$

$$-0.33333\, s_2 - 0.1666667 = \xi g \left\{ 0.66667\, p_2 + 3 r_2 - 1.111111 \right\}$$

$$1.20513\, s_2 + 0.166667 = \xi g \left\{ -0.66667\, p_2 + 0.538462\, r_2 - 1.82906 \right\}$$

$$1.53846\, s_2 + 0.50000 = \xi g \left\{ -2 p_2 - 3.461538\, r_2 - 1.39615 \right\}$$

$$s_2 + 0 = \xi g \left\{ -3 r_2 + 1.33333 \right\}$$

$$-s_2 - 0.5000 = \xi g \left\{ 2 p_2 + 9 r_2 - 3.33333 \right\}$$

$$s_2 + 0.138298 = \xi g \left\{ -0.553191\, p_2 + 0.446808\, r_2 - 1.51773 \right\}$$

$$s_2 + 0.325 = \xi g \left\{ -1.3\, p_2 - 2.25000\, r_2 - 0.907498 \right\}$$

$$2 \xi g\, p_2 + 6 \xi g\, r_2 = 2 \xi g - 0.5000$$

$$1.446809\, \xi g\, p_2 + 9.446808\, \xi g\, r_2 = 4.85106\, \xi g - 0.361702$$

$$0.74681\, \xi g\, p_2 + 2.696808\, \xi g\, r_2 = 0.61023\, \xi g - 0.186702$$

$$\xi g\, p_2 + 3\xi g\, r_2 = \xi g - 0.250000$$

$$\xi g\, p_2 + 6.52941\, \xi g\, r_2 = 3.35294\, \xi g - 0.25000$$

$$\xi g\, p_2 + 3.61111\, \xi g\, r_2 = 0.817116\, \xi g - 0.25000$$

$$\left.\begin{aligned} 3.52941\, \xi g\, r_2 &= 2.35294\, \xi g \\ 2.91830\, \xi g\, r_2 &= 2.53562\, \xi g \end{aligned}\right\}\ \text{Impossible}$$

Method of Least Square:

$$\boxed{q_2 + 0.507692\, s_2 - 0.0166667 = \xi g \{ 0.066667\, p_2 + 0.707692\, r_2 - 0.441281 \}}$$

$$\cancel{1.5000\, q_2 + s_2 - 0.12500 = \xi g \{ 0.50000\, p_2 + 0.083333 \}}$$

$$\cancel{3\, r_2 + s_2 + 0.25000 = \xi g \{ - p_2 - 3 r_2 + 0.833333 \}}$$

$$\cancel{0.65\, q_2 + s_2 + 0.1625 = \xi g \{ -0.65\, p_2 + 0.3000\, r_2 - 1.00833 \}}$$

$$\boxed{\cancel{q_2 + 0.582525\, s_2 + 0.0558253 = \xi g \{ -0.223301\, p_2 - 0.640778\, r_2 - 0.0177194 \}}}$$

$$0.666667\, q_2 + 0.444444\, s_2 - 0.055556 = \xi g \{0.222222\, p_2 + 0.0370371\}$$

$$0.333333\, q_2 + 0.111111\, s_2 + 0.0277777 = \xi g \{-0.111111\, p_2 - 0.33333\, r_2 + 0.0925926\}$$

$$1.53846\, q_2 + 2.36686\, s_2 + 0.384615 = \xi g \{-1.53846\, p_2 - 0.710058\, r_2 - 2.38658\}$$

$$2.53846\, q_2 + 2.92241\, s_2 + 0.356836 = \xi g \{-1.42735\, p_2 - 1.043391\, r_2 - 2.25695\}$$

$$\boxed{q_2 + 1.15125\, s_2 + 0.140573 = \xi g \{-0.562289\, p_2 - 0.411032\, r_2 - 0.889101\}}$$

$$0.333333\, q_2 + 0.222222\, s_2 - 0.0277777 = \xi g \{0.111111\, p_2 + 0.0185185\}$$

$$0.33333\, q_2 \qquad - 0.0277777 = \xi g \{0.111111\, p_2 + 0.666667\, r_2 - 0.277777\}$$

$$-0.33333\, q_2 - 0.111111\, s_2 - 0.0277777 = \xi g \{0.111111\, p_2 + 0.33333\, r_2 - 0.0925926\}$$

$$-\; q_2 - 1.53846\, s_2 - 0.25000 = \xi g \{p_2 + 0.461538\, r_2 + 1.55128\}$$

$$q_2 \qquad - 0.25000 = \xi g \{p_2 + 3\, r_2 - 0.155128\}$$

$$0.33333\, q_2 - 1.42735\, s_2 - 0.583333 = \xi g \{2.33333\, p_2 + 4.461538\, r_2 + 1.04430\}$$

$$\boxed{q_2 - 4.28205\, s_2 - 1.75000 = \xi g \{7\, p_2 + 13.3846\, r_2 + 3.13290\}}$$

$$2q_2 + 0 - 0.1666667 = \frac{\xi}{g}\left\{0.666667\,p_2 + 4\,r_2 - 1.666667\right\}$$

$$-q_2 - 0.33333\,s_2 - 0.083333 = \frac{\xi}{g}\left\{0.33333\,p_2 + r_2 - 0.277778\right\}$$

$$-0.461538\,q_2 - 0.710058\,s_2 - 0.115385 = \frac{\xi}{g}\left\{0.461538\,p_2 + 0.213017\,r_2 + 0.715975\right\}$$

$$3q_2 - 0.75000 = \frac{\xi}{g}\left\{3p_2 + 9r_2 - 0.465384\right\}$$

$$3.538462\,q_2 - 1.043391\,s_2 - 1.115385 = \frac{\xi}{g}\left\{4.461538\,p_2 + 14.213017\,r_2 - 1.693854\right\}$$

$$\boxed{q_2 - 0.294871\,s_2 - 0.315218 = \frac{\xi}{g}\left\{1.26087\,p_2 + 4.01672\,r_2 - 0.478698\right\}}$$

The equations for constants are then

$$q_2 + 0.507692\, s_2 - 0.016667\, \xi g\, p_2 - 0.707692\, \xi g\, r_2 = -0.441281\, \xi g + 0.0166667$$

$$q_2 + 1.15125\, s_2 + 0.562289\, \xi g\, p_2 + 0.411032\, \xi g\, r_2 = -0.889131\, \xi g - 0.140573$$

$$q_2 - 4.28205\, s_2 - 7.00000\, \xi g\, p_2 - 13.3846\, \xi g\, r_2 = 3.13290\, \xi g + 1.250000$$

$$q_2 - 0.294871\, s_2 - 1.26087\, \xi g\, p_2 - 4.01672\, \xi g\, r_2 = -0.478698\, \xi g + 0.315218$$

$$0.64356\, s_2 + 0.628956\, \xi g\, p_2 + 1.118724\, \xi g\, r_2 = -0.447820\, \xi g - 0.157240$$

$$5.43330\, s_2 + 7.562289\, \xi g\, p_2 + 13.79563\, \xi g\, r_2 = -4.02200\, \xi g - 1.890573$$

$$3.98718\, s_2 + 5.73913\, \xi g\, p_2 + 9.36788\, \xi g\, r_2 = -3.61160\, \xi g - 1.43478$$

$$s_2 + 0.977310\, \xi g\, p_2 + 1.73834\, \xi g\, r_2 = -0.695850\, \xi g - 0.244329$$

$$s_2 + 1.391839\, \xi g\, p_2 + 2.53909\, \xi g\, r_2 = -0.740249\, \xi g - 0.347960$$

$$s_2 + 1.439397\, \xi g\, p_2 + 2.34950\, \xi g\, r_2 = -0.905804\, \xi g - 0.359849$$

$$0.462087\, \xi g\, p_2 + 0.61116\, \xi g\, r_2 = -0.209954\, \xi g - 0.115520$$

$$0.414529\, \xi g\, p_2 + 0.80075\, \xi g\, r_2 = -0.044399\, \xi g - 0.103631$$

$$\xi g\, p_2 + 1.32261\, \xi g\, r_2 = -0.454359\, \xi g - 0.249996$$

$$\xi g\, p_2 + 1.93171\, \xi g\, r_2 = -0.107107\, \xi g - 0.249996$$

$$0.60910\, \xi g\, r_2 = 0.347252\, \xi g$$

$$\boxed{\xi_g r_2 = 0.570106\, \xi_g}$$

$$2\, \xi_g\, p_2 = -2.41678\, \xi_g - 2 \times 0.25000$$

$$\boxed{\xi_g\, p_2 = -1.20839\, \xi_g - 0.25000}$$

$$3 s_2 = (4.60233 - 3.77805 - 2.34190)\xi_g + 0.952163 - 0.952138$$

$$\boxed{s_2 = -0.505873\, \xi_g}$$

$$4 q_2 = \xi_g(-1.47613 - 9.38345 + 10.08972 + 1.32382) - 1.94131 + 1.94131$$

$$\boxed{q_2 = 0.138490\, \xi_g}$$

Check: ξ_g

$$\begin{array}{r} +0.138490 \\ -0.256828 \\ +0.080559 \\ -0.403459 \\ \hline -0.441238 \end{array} \qquad + 0.0166667$$

O. K.

The extensional strain energy in the circular region

$$\widehat{rr}+\widehat{\theta\theta} = \frac{Ef}{64}\left[\left\{Q_0 + 16\frac{a^2}{R^2}\left(1 - f\frac{a^2}{R^2}\right)\left(\frac{a}{R}\right)^2 + 8\left(2f\frac{a^2}{R^2} - 1\right)\left(\frac{a}{R}\right)^4 - \frac{16}{3}f\left(\frac{a}{R}\right)^6\right\}\right.$$

$$\left. + \cos 2\theta\left\{12R_2\left(\frac{a}{R}\right)^2 - \frac{16}{3}\left(\frac{a}{R}\right)^4\right\}\right]$$

$$-\left\{2P_2 + \frac{1}{3}\left(\frac{a}{R}\right)^4\right\}\left\{2P_2 + 12R_2\left(\frac{a}{R}\right)^2 - 5\left(\frac{a}{R}\right)^4\right\} - \left\{2P_2 + 6R_2\left(\frac{a}{R}\right)^2 - \frac{5}{3}\left(\frac{a}{R}\right)^4\right\}^2$$

$$= -\left[4P_2^2 + \cancel{\frac{2}{3}P_2\left(\frac{a}{R}\right)^4} + 24P_2\cancel{R_2}\left(\frac{a}{R}\right)^2 + \cancel{4R_2\left(\frac{a}{R}\right)^6} - \cancel{10P_2}\left(\frac{a}{R}\right)^4 - \frac{5}{3}\left(\frac{a}{R}\right)^6\right]$$

$$-\left[4P_2^2 + \cancel{36R_2^2\left(\frac{a}{R}\right)^4} + \frac{25}{9}\left(\frac{a}{R}\right)^8 + 24P_2\cancel{R_2}\left(\frac{a}{R}\right)^2 - \frac{20}{3}P_2\left(\frac{a}{R}\right)^4 - \cancel{20}R_2\left(\frac{a}{R}\right)^6\right]$$

$$= -\left[8P_2^2 + 36R_2^2\left(\frac{a}{R}\right)^4 + 48P_2R_2\left(\frac{a}{R}\right)^2 - 16P_2\left(\frac{a}{R}\right)^4 - 16R_2\left(\frac{a}{R}\right)^6 + \frac{10}{9}\left(\frac{a}{R}\right)^8\right]$$

$$+ 2(1+\nu)\left[4P_2^2\left(\frac{a}{R}\right)^2 + 6R_2^2\left(\frac{a}{R}\right)^6 + 12P_2R_2\left(\frac{a}{R}\right)^4 - \frac{8}{3}P_2\left(\frac{a}{R}\right)^6 - 2R_2\left(\frac{a}{R}\right)^8 + \frac{1}{9}\left(\frac{a}{R}\right)^{10}\right]$$

$$+ \left[24R_2^2\left(\frac{a}{R}\right)^6 - 16R_2\left(\frac{a}{R}\right)^8 + \frac{16\times 1.6}{3}\left(\frac{a}{R}\right)^{10}\right]$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[s_2^2 + 2(1+\nu)\left\{(r_0^2+2s_2^2)+12 s_2 q_2 + 12 q_2^2\right\}\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2\Big[\left\{q_0^2+\frac{8}{3}(4-3g)q_0 + 34.1333 - 49.7778g + 26.4127g^2\right\}$$

$$-(1+\nu)\left\{\frac{1}{2}q_0^2 + 4\left(\frac{4}{3}-g\right)q_0 + 14.2222 - 14.2222g - 6.2222g^2\right\}$$

$$+\left\{24 r_2^2 - 16 r_2 + \frac{25.6}{3}\right\}$$

$$+(1+\nu)\left\{8p_2^2 + 12 r_2^2 + 24 p_2 r_2 - \frac{16}{3}p_2 - 4 r_2 + \frac{2}{9}\right\}\Big]$$

$$\frac{\mathcal{E}_3}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \quad 0.122100\,\frac{g^2}{K^2}$$

$$\frac{p}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{2(1-\nu)s_2 - 4\nu r_0\right\}$$

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.50587^2\xi^2 g^2 + 2.6\Big\{\xi^2 g^2\frac{4}{9}(g^2-4g+4) + 2\times 0.50587^2\xi^2 g^2$$

$$-12\times 0.13849\times 0.50587\,\xi^2 g^2 + 12\times 0.13849^2\xi^2 g^2\Big\}\Big]$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\xi^2 g^2\Big(\begin{matrix}1.58660\\ 1.77777\\ -0.84070\\ +0.23015\end{matrix}\quad - 1.77777g \quad + 0.44744g^2\Big)\Big]$$

Energy outside in circle

$$\frac{\mathcal{E}_1}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\xi^2 g^2\left(0.44444g^2 - 1.77776g + 2.75383\right)\right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \xi^2 g^2 \Big[\Big\{0.35\, g_0^2 + (3.7333 - 2.80000\, g)\, g_0$$

$$+ 15.6445 - 31.2890\, g + 34.5015\, g^2\Big\}$$

$$+ \Big\{10.4\, p_2^2 + 39.6\, r_2^2 + 31.2\, p_2 r_2 - 6.93333\, p_2 - 21.2\, r_2 + 8.82222\Big\}\Big]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2 \Big[\, 0.35\big(1 - 8(\xi g) + 5.33333(\xi g) g\big)^2$$

$$+ \xi g\,(3.7333 - 2.80000\, g)\big(1 - 8(\xi g) + 5.3333(\xi g) g\big)$$

$$+ (15.6445 - 31.2890\, g + 34.5015\, g^2)\,\xi^2 g^2$$

$$+ 10.4 \times (1.20839\,\xi g + 0.25000)^2 + 39.6 \times 0.570106\, \xi^2 g^2 - 31.2 \times 0.570106\, \xi g \times (1.20839\,\xi g + 0.25000)$$

$$+ 6.93333\, \xi g\,(1.20839\,\xi g + 0.25000) - 21.2\, \xi g \times 0.570106\, \xi g + 8.82222\, \xi^2 g^2\Big\}\Big]$$

$\xi^2 g^2 \Big[$

g^2 9.95555	g −29.8666	+22.40000
−14.93333	+42.3111	−29.86666
+34.5015	−31.2890	+15.6445
		+15.1862
		+22.5762
		−21.4940
		+8.3782
		−12.0862
		+8.8222

$$(\xi g)\left[\ g \quad \begin{matrix} 3.73333 \\ -2.80000 \end{matrix} \qquad \begin{matrix} -\ 5.6 \\ +\ 3.73333 \\ +\ 6.28363 \\ -\ 4.44683 \\ +\ 1.73333 \end{matrix}\ \right]$$

$$+\left[\ 0.35 + 0.65\right]$$

$$\boxed{\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\ \xi^2 g^2\left(29.5237\, g^2 - 18.8445 + 29.5605\right) + \xi g\left(0.93333\, g + 1.70346\right)\right].}$$

$$\frac{\mathcal{V}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a^2}{R^2}\right)\left\{-1.4 \times 0.50587\,(\xi g) - 1.2 \times 0.66167\,(g-2)\,\xi g\right\}$$

$$\boxed{\frac{\mathcal{V}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{0.89178 - 0.80000\, g\right\}(\xi g)}$$

$$\frac{\mathcal{E}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{\ \xi^2 g^2\left(29.9681\, g^2 - 20.6223\, g + 32.3143\right) + \xi g\left(1.73333\, g + 0.81168\right) + 0.122100\,\frac{g^2}{K^2}\right\}$$

Let σ be compression,

$$K^2 = \frac{0.122100\, g}{\xi\,(1.73333\, g + 0.81168) - \xi^2 (29.9681 g^3 - 20.6223 g^2 + 32.3143 g)}$$

$$\xi = \frac{1}{2}\, \frac{1.73333 g + 0.81168}{g\,(29.9681 g^2 - 20.6223 g + 32.3143)} = \frac{1}{64} \frac{1}{\left(\frac{t}{R}\right)} \frac{1}{K} \left(\frac{a}{R}\right)^2$$

$$K^2 = \frac{0.488400\, g^2 (29.9681 g^2 - 20.6223 g + 32.3143)}{(1.7333\, g + 0.81168)^2}$$

g = amplitude factor

When $g = \frac{0.89178}{0.8000} = 1.1147$,

$$K^2 = \frac{0.4884 \times 1.2426 \times (37.238 - 22.988 + 32.3143)}{(0.93335}$$

$g = 0.1$

$$K = 0.1 \frac{\sqrt{0.488400\,(30.5518)}}{0.98501} = \underline{\underline{0.3920}}$$

$$\left(\frac{a}{R}\right)^2 = 32 \left(\frac{t}{R}\right) \frac{\sqrt{0.488400}}{\sqrt{29.9681 g^2 - 20.6223 g + 32.3143}} = 4.04 \left(\frac{t}{R}\right)$$

$\left(\frac{t}{R}\right) = \frac{1}{1000}$, $\left(\frac{a}{R}\right) = 0.0636$

at $g = 0.1$

$$\xi^2 g^2 = \frac{1}{4} \frac{(1.73333g + 0.81168)^2}{(29.9681g^2 - 20.6223g + 32.3143)^2} = 0.00025985$$

$$\xi g = \frac{1}{2} \frac{1.7333g + 0.81168}{29.9681g^2 - 20.6223g + 32.3143} = \frac{1}{2} \frac{0.98501}{30.5518} = 0.016120$$

$$\mathcal{E}_1 \sim 0.00025985 \times 2.58049 = \underline{0.00067054} + \quad \text{Energy outside the circular region}$$

$$\mathcal{E}_2 \sim 0.00025985 \times 27.9713 - 0.016120 \times 1.79679$$

$$= 0.007268 - 0.028964 = \underline{-0.021696} \quad \text{Energy — extensional in the region}$$

$$\mathcal{E}_3 \sim 0.122100 \frac{0.01}{0.3920^2} = \underline{+0.00795} \quad \text{Bending Energy}$$

$$\mathcal{P} \sim -0.016120 \times 0.81178 = \underline{-0.013086} \ (?) \quad \text{Increase in potential}$$

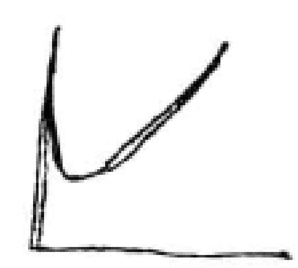
$$\frac{w_{max}}{t} = f \frac{1}{4} \left(\frac{a}{R}\right)^4 \left(\frac{R}{t}\right) = \frac{g}{4} \left(\frac{a}{R}\right)^2 \left(\frac{R}{t}\right) = \frac{4.04}{4} g = 0.101 \quad (\text{Too small})$$

For virtual work

$$\frac{P}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{-2(1+\nu)\eta_3 + 4\delta_2\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{-\xi g \times 2.6 \times 0.166667(g-2) - 2.02349\,\xi g\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{1.44318 - 1.73333\,g\right\}\xi g$$



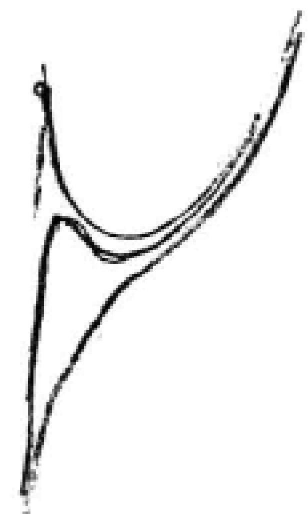
$$K^2 = \frac{0.4884\,g\,(39.9681\,g^2 - 20.6223\,g + 32.3143)}{(2.6666\,g + 0.26028)^2}$$

$k = 1.480$

$g = 0.1$ $\quad k = \frac{0.0699\sqrt{30.55}}{0.52[illegible]} = 1.7[illegible]$

$g = 0.05$ $\quad k = \frac{0.03495}{0.39361}\sqrt{31.36} = 0.498$

$g = 0.15$ $\quad k = \frac{0.1048\sqrt{29.914}}{0.6598} = 0.870$



~~Take~~ $\cancel{2.91830\,\xi g r_2 = 2.53582\,\xi g}$

~~or~~ $\boxed{\cancel{\xi g r_2 = 0.86894\,\xi g}}$

If we drop the condition of continuity of u,

$$q_2 + 0.66667\,s_2 - 0.083333 = \xi g\{0.33333\,p_2 + 0.055556\}$$
$$q_2 \qquad - 0.083333 = \xi g\{0.3333\,p_2 + 2r_2 - 0.83333\}$$
$$-q_2 - 0.33333\,s_2 - 0.083333 = \xi g\{0.33333\,p_2 + r_2 - 0.277778\}$$
$$q_2 \qquad - 0.250000 = \xi g\{p_2 + 3r_2 - 0.155198\}$$

$$0.666667\,s_2 + 0 = \xi g\{-2r_2 + 0.888889\}$$
$$-0.33333\,s_2 - 0.166667 = \xi g\{0.66667\,p_2 + 3r_2 - 1.111111\}$$
$$-0.33333\,s_2 - 0.33333 = \xi g\{1.3333\,p_2 + 4r_2 - 0.432903\}$$

$$s_2 + 0 = \xi g\{-3r_2 + 1.33333\}$$
$$-s_2 - 0.5000 = \xi g\{2p_2 + 9r_2 - 3.33333\}$$
$$-s_2 - 1.000 = \xi g\{4p_2 + 12r_2 - 1.298709\}$$

$$2\xi g p_2 + 6\xi g r_2 = 2\,\xi g - 0.50000$$
$$2\xi g p_2 + 3\xi g r_2 = -2.03462\,\xi g - 0.5000$$

$$3\xi g r_2 = 4.03462\,\xi g$$

$$\boxed{\xi g r_2 = 1.34487\, \xi g}$$

$$\xi g p_2 = \frac{1}{4}\left\{(-12.1038 - 0.03462)\xi g - 1.000\right\}$$

$$\boxed{\xi g p_2 = -3.034675\, \xi g - 0.2500}$$

$$3 s_2 + 1.5000 = \xi g\left\{-6 p_2 - 24 r_2 + 5.96537\right\}$$

$$= \xi g\left\{18.2081 - 32.27688 + 5.96537\right\} + 1.5000$$

$$\boxed{s_2 = -2.70113\, \xi g}$$

$$4 q_2 + s_2 - 0.33333 = \xi g\left\{1.3333\, p_2 + 4 r_2 - 0.655127\right\}$$

$$q_2 = \frac{1}{4}\left\{2.70113 - 4.04623 + 5.37948 - 0.65513\right\}\xi g$$

$$\boxed{q_2 = 0.84481\, \xi g}$$

$$\frac{\mathcal{E}_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[2.70113^2+2.6\Big\{\frac{4}{9}(g^2-4g+4)+2\times 2.70113^2$$

$$-12\times 2.70113\times 0.84481+12\times 0.84481^2\Big\}\Big](\xi g)^2$$

$$=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left[\,0.44444\,g^2-1.77778\,g+\begin{array}{r}1.77778\\7.29610\\37.93972\\-\,27.38330\\+\,8.56445\end{array}\right](\xi g)^2$$

$$\frac{\mathcal{E}_1}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[\,0.44444g^2-1.77778g+28.19475\Big](\xi g)^2$$

$$\frac{\mathcal{E}_2}{R^3}=\left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[0.35\,(1-8\,\xi g+5.3333(\xi g)^2)^2+\xi g\,(3.7333-2.800\,\xi g)(1-8\,\xi g+5.333\,\xi g)$$

$$+(34.5015\,g^2-31.2890\,g+15.6445)(\xi g)^2$$

$$+10.4\,(3.03425\,\xi g+0.2500)^2+39.6\times 1.34487^2(\xi g)^2$$

$$-\quad 35.02661\,\xi g\,(3.034675\,\xi g+0.2500)-28.51124\,(\xi g)^2+8.8292\,(\xi g)^2\Big]$$

$$(\xi g)^2\left[\,29.5237\,g^2-18.8445\,g+\begin{array}{r}22.40000\\-29.86666\\+15.6445\\+95.7810\\+71.6237\\-106.2944\\-28.5112\\+8.8292\end{array}\right]$$

$$\xi g \left[\begin{matrix} 3.73333\, g \\ -2.80000 \end{matrix} \qquad \begin{matrix} -5.6 \\ 3.73333 \\ +15.7807 \\ -8.7567 \end{matrix} \right]$$

$$\frac{\mathcal{E}_2}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\Big[(\xi g)^2\left(29.5237 g^2 - 18.8445 g + 49.5991\right)$$

$$+ \xi g \left(0.93333 g + 5.1573\right)\Big]$$

$$\frac{\mathcal{P}}{R^3} = \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2\left\{-\xi g \times 2.6 \times 0.66667(g-2) - 10.8045\,\xi g\right\}$$

$$= \left(\frac{t}{R}\right)\frac{\sigma^2}{2E}\pi\left(\frac{a}{R}\right)^2(\xi g)\left\{-1.73333 g - 7.3378\right\}$$

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2}$$

$\beta = 3.8317$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{r}{R}\right)^2 \sin^2\theta}{2} - f\left(\frac{1}{2}\frac{a^2}{R^2}\right)\left\{J_0\left(\beta\frac{r}{a}\right) + \eta\right\}\frac{1}{\delta}$$

$\gamma = 0.4028$

$\delta = 1.4028$

$$\frac{1}{R}\frac{\partial w}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta}{a}J_0'\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial w_0}{\partial r} = -r\left(\frac{1}{R^2}\right)\sin^2\theta$$

$$\frac{1}{R}\frac{\partial^2 w}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta - \frac{f}{\delta}\frac{1}{2}\frac{a^2}{R^2}\frac{\beta^2}{a^2}J_0''\left(\beta\frac{r}{a}\right)$$

$$\frac{1}{R}\frac{\partial^2 w_0}{\partial r^2} = -\frac{1}{R^2}\sin^2\theta$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}(\sin^2\theta)^2 - \frac{1}{R^2}\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\frac{a\beta}{r}J_0'\left(\beta\frac{r}{a}\right)\right]\left[\sin^2\theta + \frac{1}{2}\frac{f}{\delta}\beta^2 J_0''\left(\beta\frac{r}{a}\right)\right]$$

$$= -\frac{1}{R^2}\left[\frac{1}{2}\frac{f}{\delta}\beta^2\sin^2\theta\left\{J_0'' + \frac{1}{\left(\beta\frac{r}{a}\right)}J_0'\right\} + \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r}J_0'J_0''\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right\}$$

$$= -\frac{1}{R^2}\cos 2\theta\cdot\frac{1}{2}\frac{f}{\delta}\beta^2 J_0''$$

$$\nabla^4\phi = \frac{E}{R^2}\Big[J_0 \frac{1}{2}\frac{f}{\delta}\beta^2 \sin^2\theta - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta\Big]$$ 362

$$= \frac{E}{R^2}\Big[J_0 \frac{1}{4}\frac{f}{\delta}\beta^2(1-\cos 2\theta) - \frac{1}{4}\frac{f^2}{\delta^2}\frac{a\beta^3}{r} J_0' J_0'' - \frac{1}{2}\frac{f}{\delta}\beta^2 J_0'' \cos 2\theta\Big]$$

$$= \frac{1}{4}\frac{f}{\delta}\beta^2 \frac{E}{R^2}\Big[\Big\{ J_0 - \beta^2\frac{f}{\delta}\frac{J_0' J_0''}{(\beta\frac{r}{a})}\Big\} - \cos 2\theta\Big\{J_0 + 2J_0''\Big\}$$

$$= \frac{1}{4}\frac{gE}{R^2}\Big[\Big\{J_0 - g\frac{J_0' J_0''}{z}\Big\} - \cos 2\theta\,(J_0 + 2J_0'')\Big\}$$

where $g = \frac{f}{\delta}\beta^2$, $z = (\beta\frac{r}{a})$

$$\frac{J_0' J_0''}{z} = \frac{J_1 J_1'}{z} = \frac{J_1^2}{z^2} - \frac{J_1 J_2}{z}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r+2)!\,(\frac{1}{2}z)^{2r}}{4(r!)(r+1)!\,(r+1)!\,(r+2)!} - \sum_{r=0}^{\infty}\frac{(-1)^r (2r+3)!\,(\frac{1}{2}z)^{2r+2}}{2(r!)(r+1)!\,(r+2)!\,(r+3)!}$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{z}{2}\right)^2 + \frac{5}{48}\left(\frac{z}{2}\right)^4 - \frac{7}{288}\left(\frac{z}{2}\right)^6 + \frac{7}{1920}\left(\frac{z}{2}\right)^8 - \frac{11}{28800}\left(\frac{z}{2}\right)^{10}$$

$$+ \frac{143}{4838400}\left(\frac{z}{2}\right)^{12} - \cdots\cdots$$

$$-\Big\{\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{24}\left(\frac{z}{2}\right)^4 + \frac{7}{96}\left(\frac{z}{2}\right)^6 - \frac{7}{480}\left(\frac{z}{2}\right)^8 + \frac{11}{5760}\left(\frac{z}{2}\right)^{10} + \frac{143}{806400}\left(\frac{z}{2}\right)^{12}\Big\} -$$

Thus

$$\frac{J_0' J_0''}{z} = \frac{1}{4} - \frac{1}{2}\left(\frac{z}{2}\right)^2 + \frac{5}{16}\left(\frac{z}{2}\right)^4 - \frac{7}{72}\left(\frac{z}{2}\right)^6 + \frac{7}{384}\left(\frac{z}{2}\right)^8 - \frac{11}{4800}\left(\frac{z}{2}\right)^{10} + \frac{143}{691200}\left(\frac{z}{2}\right)^{12} - \cdots$$

The particular integral for this term is

$$\frac{\phi_1}{R^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4 \left(\frac{f}{\delta}\right)^2 E \left\{ \frac{1}{16}\left(\frac{z}{2}\right)^4 - \frac{1}{72}\left(\frac{z}{2}\right)^6 + \frac{5}{2304}\left(\frac{z}{2}\right)^8 - \frac{7}{28800}\left(\frac{z}{2}\right)^{10} + \frac{7}{345600}\left(\frac{z}{2}\right)^{12} \right.$$

$$\left. - \frac{11}{4 \times 2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200 \times 784 \times 4}\left(\frac{z}{2}\right)^{16} - \cdots \right\}$$

The particular integral for the term $\frac{1}{4}\frac{gE}{R^2} J_0$ is

$$\frac{\phi_2}{R^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4 \left(\frac{f}{\delta}\right) \frac{1}{\beta^2} J_0$$

The particular integral for the term $-\frac{1}{4}\frac{gE}{R^2}\cos 2\theta\,(J_0 + 2J_0'')$ is

$$\frac{\phi_3}{R^2} = -\frac{1}{4}\left(\frac{a}{R}\right)^4 \left(\frac{f}{\delta}\right) \frac{1}{\beta^2} \cos 2\theta \; J_2$$

Then the total particular integral is

$$\frac{\phi_1 + \phi_2 + \phi_3}{R^2} = \frac{1}{4}\left(\frac{a}{R}\right)^4 \frac{1}{\beta^4}\, gE \left[\left\{ J_0 - \frac{g}{\delta f}\left[\frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{18}\left(\frac{z}{2}\right)^6 + \frac{5}{576}\left(\frac{z}{2}\right)^8 - \frac{7}{7200}\left(\frac{z}{2}\right)^{10} \right.\right.\right.$$

$$\left.\left. + \frac{7}{86400}\left(\frac{z}{2}\right)^{12} - \frac{11}{2116800}\left(\frac{z}{2}\right)^{14} + \frac{143}{691200 \times 784}\left(\frac{z}{2}\right)^{16} - \cdots \right] \right\}$$

$$\left. - J_2 \cos 2\theta \right\} = \frac{\Phi}{R^2}$$

The stresses due to this particular integral are:

$$\frac{1}{r}\frac{\partial\Phi}{\partial r} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z}\frac{\partial\Phi}{\partial z} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\varrho E}{\beta^2}\Big[\frac{J_0'}{z} - \frac{9}{64}\Big[\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{5}{288}\left(\frac{z}{2}\right)^6$$

$$- \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10} - \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \dots\Big]$$

$$- \frac{J_2'}{z}\cos 2\theta\Big\}$$

$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = \left(\frac{\beta}{a}\right)^2 \frac{1}{z^2}\frac{\partial^2\Phi}{\partial\theta^2} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\varrho E}{\beta^2}\Big[+\frac{J_2}{z^2}\,4\cos 2\theta\Big]$$

$$\widehat{rr}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\varrho E}{\beta^2}\Big[\frac{J_0'}{z} - \frac{9}{64}\Big\{\frac{1}{4}\left(\frac{z}{2}\right)^2 - \frac{1}{12}\left(\frac{z}{2}\right)^4 + \frac{5}{288}\left(\frac{z}{2}\right)^6 - \frac{7}{2880}\left(\frac{z}{2}\right)^8 + \frac{7}{28800}\left(\frac{z}{2}\right)^{10}$$

$$- \frac{11}{604800}\left(\frac{z}{2}\right)^{12} + \frac{143}{135475200}\left(\frac{z}{2}\right)^{14} - \dots\Big\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2}\right)\Big]$$

$$\widehat{\theta\theta}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\varrho E}{\beta^2}\Big[J_0'' - \frac{9}{64}\Big[\frac{3}{4}\left(\frac{z}{2}\right)^2 - \frac{5}{12}\left(\frac{z}{2}\right)^4 + \frac{35}{288}\left(\frac{z}{2}\right)^6 - \frac{63}{2880}\left(\frac{z}{2}\right)^8 + \frac{77}{28800}\left(\frac{z}{2}\right)^{10}$$

$$- \frac{143}{604800}\left(\frac{z}{2}\right)^{12} + \frac{2145}{135475200}\left(\frac{z}{2}\right)^{14} - \dots\Big] - J_2''\cos 2\theta\Big\}$$

$$\widehat{\theta r}_1 = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\varrho E}{\beta^2}\Big[2\sin 2\theta\left(\frac{J_2'}{z} - \frac{J_2}{z^2}\right)\Big]$$

$$\frac{1}{E}(\widehat{rr} - \nu\widehat{\theta\theta}) = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{g}{\beta^2}\Bigg[\left(\frac{J_0'}{z} - \nu J_0''\right) - g\Bigg\{\frac{(1-3\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{(1-5\nu)}{48}\left(\frac{z}{2}\right)^4$$

$$+ \frac{5(1-7\nu)}{288\times 4}\left(\frac{z}{2}\right)^6 - \frac{7(1-9\nu)}{2880\times 4}\left(\frac{z}{2}\right)^8 + \frac{7(1-11\nu)}{4\times 28800}\left(\frac{z}{2}\right)^{10} - \frac{11(1-13\nu)}{604800\times 4}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{143(1-15\nu)}{4\times 135475200}\left(\frac{z}{2}\right)^{14} - - - - \Bigg\} - \cos 2\theta\left\{\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2''\right\}\Bigg]$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{\partial w_0}{\partial r}\right)^2\right\} = \frac{1}{2}\left[\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{f}{\delta}J_0'\left\{\frac{1}{2}\left(\frac{a}{R}\right)\beta\frac{f}{\delta}J_0' + 2\left(\frac{a}{R}\right)\sin^2\theta\right\}\right]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\left(zJ_0' + g\frac{J_0'^2}{2}\right) - zJ_0'\cos 2\theta\right]$$

$$J_0'^2 = J_1^2 = \sum_n \frac{(-1)^n (2n+2)!\left(\frac{z}{2}\right)^{2n+2}}{n!\,(n+2)!\,(n+1)!\,(n+1)!}$$

$$= \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^4 + \frac{5}{12}\left(\frac{z}{2}\right)^6 - \frac{7}{72}\left(\frac{z}{2}\right)^8 + - - -$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\Bigg[\left\{\left(\frac{J_0'}{z} - zJ_0'\right) - \nu J_0''\right\} - g\Bigg\{\frac{3(3-\nu)}{16}\left(\frac{z}{2}\right)^2 - \frac{5(5-\nu)}{48}\left(\frac{z}{2}\right)^4$$

$$+ \frac{5\times 7(7-\nu)}{1152}\left(\frac{z}{2}\right)^6 - \frac{7\times 9(9-\nu)}{11520}\left(\frac{z}{2}\right)^8 + \frac{7\times 11(11-\nu)}{115200}\left(\frac{z}{2}\right)^{10} - \frac{11\times 13(13-\nu)}{2419200}\left(\frac{z}{2}\right)^{12}$$

$$+ \frac{143\times 15(15-\nu)}{541900800}\left(\frac{z}{2}\right)^{14} - - - \Bigg\} - \cos 2\theta\left(\frac{J_2'}{z} - \frac{4J_2}{z^2} - \nu J_2'' - zJ_0'\right)\Bigg]$$

$$\frac{u_1}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^3\frac{1}{\beta^3}\frac{q}{\gamma}\left[\left\{\left(\quad zJ_0''\quad\right)-\nu J_0'\right\}-\frac{q}{\gamma}\left\{\frac{(3-\nu)}{8}\left(\frac{z}{2}\right)^3-\frac{(5-\nu)}{24}\left(\frac{z}{2}\right)^5\right.\right.$$

$$+\frac{5(7-\nu)}{576}\left(\frac{z}{2}\right)^7-\frac{7(9-\nu)}{5760}\left(\frac{z}{2}\right)^9+\frac{7(11-\nu)}{57600}\left(\frac{z}{2}\right)^{11}-\frac{11(13-\nu)}{1209600}\left(\frac{z}{2}\right)^{13}$$

$$\left.\left.+\frac{143(15-\nu)}{270950400}\left(\frac{z}{2}\right)^{15}-\cdots\right\}-\cos 2\theta\left\{-\nu J_2'-J_2'-zJ_1'-J_0'\right\}\right.$$

$$\int\left(\frac{J_2'}{z}-\frac{4J_0}{z^2}-zJ_0'\right)dz=\int\left\{\frac{J_2'}{z}-\left(J_2''+\frac{J_2'}{z}+J_2\right)-zJ_0'\right\}dz$$

$$=-J_2'-\int\left(J_2+zJ_0'\right)dz$$

$$=-J_2'-\int\left\{\frac{2J_1}{z}-J_0(z)+zJ_0'\right\}dz=-J_2'+\int\left(\frac{J_0'}{z}-zJ_0'\right)dz$$

$$+\int\left(\frac{J_0'}{z}+J_0\right)dz=-J_2'+zJ_0''+\int\left(\frac{J_0'}{z}+J_0\right)dz=-J_2'+zJ_0''-J_0'$$

But
$$J_0=J_2-2J_0''$$
$$\frac{J_0'}{z}=-J_2+J_0''$$
$$J_0+\frac{J_0'}{z}=-J_0''$$

$$\frac{u}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{a}{\beta^2}\left[\left\{J_0''-\nu\frac{J_0'}{z}\right\}-\frac{q}{\gamma}\left\{\frac{(3-\nu)}{16}\left(\frac{z}{2}\right)^2-\frac{(5-\nu)}{48}\left(\frac{z}{2}\right)^4+\cdots-\right\}\right.$$

$$\left.+\cos 2\theta\left\{\frac{J_2'}{z}+J_1'+\frac{J_0'}{z}+\nu\frac{J_2'}{z}\right\}\right]$$

367

$$-\frac{u}{r}+\frac{1}{E}\left(\widehat{\theta\theta}_1-\widehat{rr}_1\right)=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{-J_2''+\cancel{\frac{\nu J_2'}{z}}-\frac{4\nu J_2}{z^2}-\frac{J_2'}{z}-J_1'-\frac{J_0'}{z}-\frac{\nu J_2''}{z}\right\}\right]$$

$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{-J_2''-\frac{J_2'}{z}-J_1'+\frac{J_1}{z}-\frac{4\nu J_2}{z^2}\right\}\right]$$

$$=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{g}{\beta^2}\left[\cos 2\theta\left\{J_2-\frac{4(1+\nu)J_2}{z^2}+\frac{J_1}{z}-J_1'\right\}\right]$$

$$\boxed{\frac{V_1}{R}=\frac{1}{4}\left(\frac{a}{R}\right)^3\frac{g}{\beta^3}\left[\frac{\sin 2\theta}{2}\left\{-zJ_2''-J_2'-zJ_1'+J_1-\frac{4\nu J_2}{z}\right\}\right]}$$

The total stress component can be expressed as

$$\widehat{rr}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\frac{1}{2}Q_0-\frac{J_1}{z}-\frac{g}{4}\left\{\frac{1}{4}\left(\frac{z}{2}\right)^2-\frac{1}{12}\left(\frac{z}{2}\right)^4+\frac{5}{288}\left(\frac{z}{2}\right)^6-\frac{7}{2880}\left(\frac{z}{2}\right)^8+\frac{7}{28800}\left(\frac{z}{2}\right)^{10}\right.\right.$$
$$\left.\left.-\frac{11}{604800}\left(\frac{z}{2}\right)^{12}+\frac{143}{135475200}\left(\frac{z}{2}\right)^{14}-\cdots\right\}-\cos 2\theta\left\{2P_2+\frac{J_1}{z}-\frac{6J_2}{z^2}\right\}\right]$$

$$\widehat{\theta\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\frac{1}{2}Q_0-\frac{1}{2}(J_0-J_2)-\frac{g}{4}\left\{\frac{3}{4}\left(\frac{z}{2}\right)^2-\frac{5}{12}\left(\frac{z}{2}\right)^4+\frac{35}{288}\left(\frac{z}{2}\right)^6-\frac{63}{2880}\left(\frac{z}{2}\right)^8\right.\right.$$
$$+\frac{27}{28800}\left(\frac{z}{2}\right)^{10}-\frac{143}{604800}\left(\frac{z}{2}\right)^{12}+\frac{2145}{135475200}\left(\frac{z}{2}\right)^{14}-\cdots\Bigg]+\cos 2\theta\left\{2P_2+12R_2z^2\right.$$
$$\left.\left.-\left(\frac{6}{z^2}-1\right)J_2+\frac{J_1}{z}\right\}\right]$$

$$\widehat{r\theta}=\frac{1}{4}\left(\frac{a}{R}\right)^2\frac{gE}{\beta^2}\left[\sin 2\theta\left\{2P_2+6R_2z^2-\frac{6J_2}{z^2}+\frac{2J_1}{z}\right\}\right]$$

The total deflection is

$$\frac{u}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{q}{\beta^3}\Big[\frac{(1-\nu)}{2} Q_0 z + J_1 - zJ_0 + \nu J_1 - \frac{g}{4}\Big\{ \frac{(3-\nu)}{2}\left(\frac{z}{2}\right)^3 - \frac{(5-\nu)}{6}\left(\frac{z}{2}\right)^5$$

$$+ \frac{5(7-\nu)}{144}\left(\frac{z}{2}\right)^7 - \frac{7(9-\nu)}{1440}\left(\frac{z}{2}\right)^9 + \frac{7(11-\nu)}{14400}\left(\frac{z}{2}\right)^{11} - \frac{11(13-\nu)}{302400}\left(\frac{z}{2}\right)^{13}$$

$$+ \frac{143(15-\nu)}{67737600}\left(\frac{z}{2}\right)^{15} - \cdots \Big\} - \cos 2\theta \Big\{ 2(1+\nu) P_2 z + 4\nu R_2 z^3 + J_1 - zJ_1'$$

$$- (1+\nu) J_2' \Big\} \Big]$$

$1.6159 = \beta^2$

$$\frac{v}{R} = \frac{1}{4}\left(\frac{a}{R}\right)^3 \frac{q}{\beta^3}\Big[\sin 2\theta \Big\{ 2(1+\nu) P_2 z + 6(1+\nu) R_2 z^3 - \frac{zJ_2''}{2} - \frac{J_2'}{2} - \frac{zJ_1'}{2}$$

$$+ \frac{1}{g} J_1 - \frac{2\nu J_2}{z} \Big\} \Big]$$

At $r = a$, $z = \beta$ $\quad \frac{\beta}{2} = \frac{3.8317}{2} = 1.6159$

$$M_a - \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{qE}{\beta^2}\Big[\frac{1}{2} Q_0 - \frac{g}{4}\Big\{ 0.25 \times 1.6159^2 - 0.083333 \times 1.6159^4 + 0.00173611 \times 1.6159^6$$

$$- 0.0002430556 \times 1.6159^8 + 0.0000243056 \times 1.6159^{10} - 0.0000181876 \times 1.6159^{12}$$

$$+ 0.0000010555 4 \times 1.6159^{14} - \cdots \Big\} - \cos 2\theta \Big\{ 2P_2 - 6 \frac{0.4025}{3.8317^2} \Big\} \Big]$$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{qE}{\beta^2}\Big[\frac{1}{2} Q_0 - \frac{g}{4}\Big\{ 0.25 \times 2.6111 - 0.083333 \times 6.8178 + 0.00173611 \times 17.8020$$

$$- 0.0002430556 \times 46.4828 + 0.0000243056 \times 121.37 - 0.0000181876 \times 316.91$$

$$+ 0.00000105554 \times 827.48 - \cdots \Big\} - \cos 2\theta \Big\{ 2P_2 - 0.1645 \Big\} \Big]$$

+0.65278	0.100	+1.95834	-0.824
-0.56815		-2.84075	
+0.03091		+0.21637	
-0.01130		-0.10170	
+0.00295		+0.03245	
-0.00576		-0.07486	
+0.00087		+0.01305	

$$\hat{N}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\sigma E}{\beta^2}\left[\frac{1}{2}Q_0 - 0.02509 - \cos 2\theta\left(2P_2 - 0.1645\right)\right]$$

$$\hat{M}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\sigma E}{\beta^2}\left[\frac{1}{2}Q_0 + 0.4027 + 0.2469 + \cos 2\theta\left\{2P_2 + 176.18 R_2 + 0.2340\right\}\right]$$

$$\hat{N}_a = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\sigma E}{\beta^2}\left[\sin 2\theta\left\{2P_2 + 88.09 R_2 - 0.1645\right\}\right]$$

The non-uniform portion of $\frac{u}{R_a}$

$$= \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\sigma E}{\beta^2}\left[-\cos 2\theta\left\{2(1+\nu)P_2 + 58.7292\nu R_2 + 0.4027 + (1+\nu)0.05483\right\}\right]$$

$$\frac{v}{R_a} = \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\sigma E}{\beta^2}\left[\sin 2\theta\left\{2(1+\nu)P_2 + 88.0914(1+\nu)R_2 + 0.3478 - 0.054834\right\}\right]$$

Put $\qquad \frac{1}{4}\left(\frac{a}{R}\right)^2 \frac{\sigma E}{\beta^2} = \eta$

Then the stress + displacement conditions give

$$\frac{1}{2}+r_0=\eta g\left\{\frac{1}{2}Q_0-0.0250g\right\} \qquad (1)$$

$$\frac{1}{2}-r_0=\eta g\left\{\frac{1}{2}Q_0+0.4027+0.206g\right\} \qquad (2)$$

$$(3)\quad \frac{1}{2}-6q_2-4s_2=\eta g\left\{0.1645-2P_2\right\}$$

$$(4)\quad 6q_2-\frac{1}{2} \qquad =\eta g\left\{2P_2+176.18R_2+0.2380\right\}$$

$$(5)\quad \frac{1}{2}+6q_2+2s_2=\eta g\left\{0.1645-2P_2-88.09R_2\right\}$$

$$(6)\quad \frac{1}{2}(1+\nu)+2(1+\nu)q_2+4s_2=\eta g\left\{-2(1+\nu)P_2-58.7276\,\nu R_2-0.4027-(1+\nu)\,\nu\,0.5483\right\}$$

$$(7)\quad 2(1+\nu)q_2-\frac{1}{2}(1+\nu)=\eta g\left\{2(1+\nu)P_2+88.0914(1+\nu)R_2+0.3426-0.05443\nu\right\}$$

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from (1) + (2) $\qquad 1=\eta g\left\{Q_0+0.4027+0.181g\right\}$

$$\boxed{Q_0=\frac{1}{\eta g}-(0.4027+0.181g)}$$

$$r_0=\eta g\left\{-(0.20135+0.0905g)-0.0250g\right\}=-\eta g\left\{0.20135+0.1155g\right\}$$

$$\boxed{r_0=-\eta g\,(0.2014+0.1155g)}$$

$$q_2 + 0.6667\,S_2 - 0.08333 = \frac{\eta g}{l}\left\{0.3333\,P_2 - 0.02742\right\}$$

$$q_2 + 0 - 0.08333 = \frac{\eta g}{l}\left\{0.3333\,P_2 + 29.36\,R_2 + 0.03967\right\}$$

$$q_2 + 0.3333\,S_2 + 0.08333 = \frac{\eta g}{l}\left\{-0.3333\,P_2 - 14.68\,R_2 + 0.02742\right\}$$ 15.29

$$q_2 + 1.5385\,S_2 + 0.2500 = \frac{\eta g}{l}\left\{-P_2 - 6.7760\,R_2 - 0.1823\right\}$$

$$q_2 + 0 - 0.2500 = \frac{\eta g}{l}\left\{P_2 + 44.0457\,R_2 + 0.1275\right\}$$ 1338 0061

$$0.6667\,S_2 + 0 = \frac{\eta g}{l}\left\{-29.36\,R_2 - 0.06709\right\}$$

$$0.3333\,S_2 + 0.16667 = \frac{\eta g}{l}\left\{-0.6667\,P_2 - 44.04\,R_2 - 0.01225\right\}$$

$$1.2052\,S_2 + 0.16667 = \frac{\eta g}{l}\left\{-0.6667\,P_2 + 7.904\,R_2 - 0.2097\right\}$$

$$1.5385\,S_2 + 0.5000 = \frac{\eta g}{l}\left\{-2\,P_2 - 50.83\,R_2 - 0.3098\right\}$$

$$S_2 + 0 = \frac{\eta g}{l}\left\{-44.04\,R_2 - 0.1006\right\}$$

$$S_2 + 0.5000 = \frac{\eta g}{l}\left\{-2\,P_2 - 132.12\,R_2 - 0.03675\right\}$$

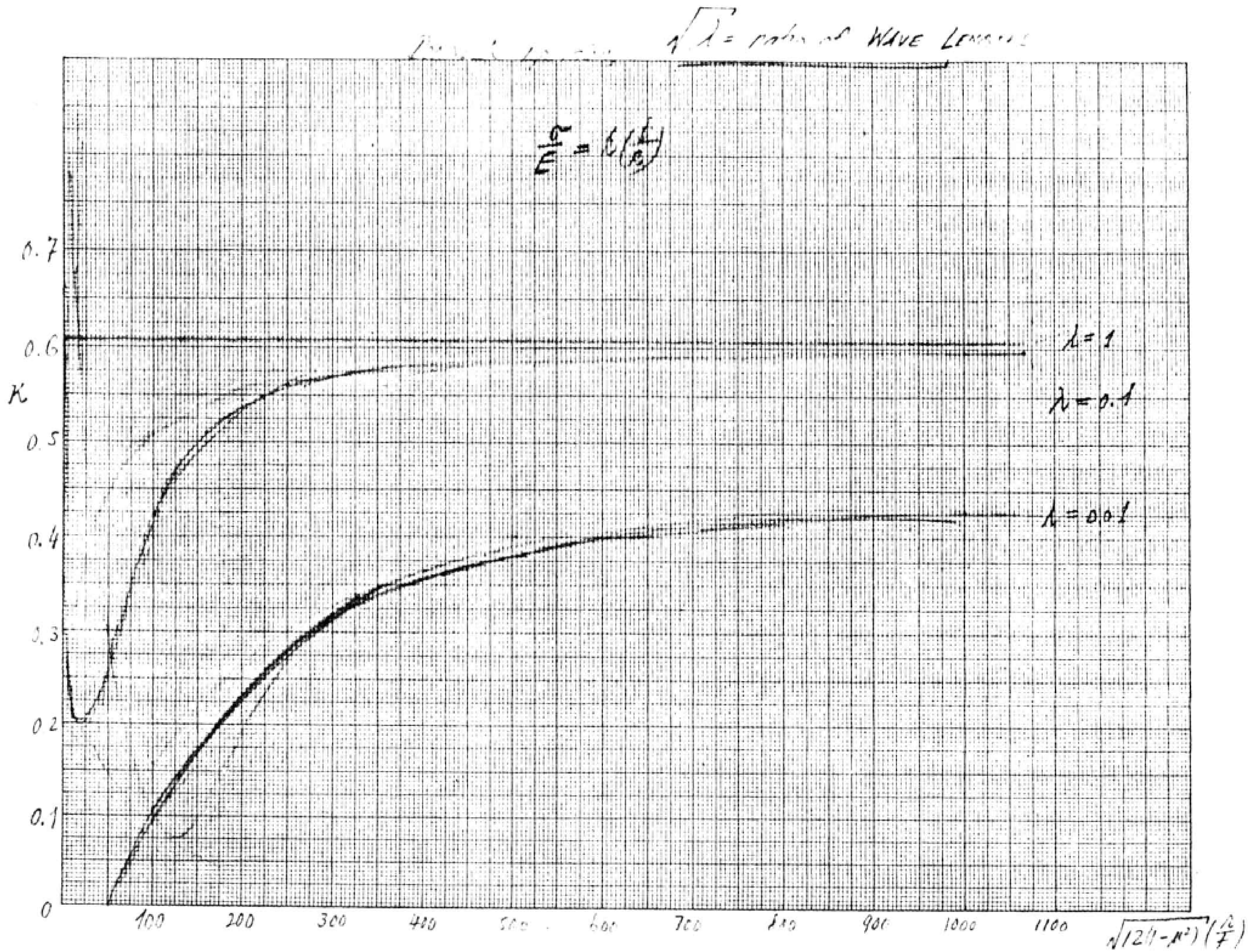
$$S_2 + 0.1383 = \frac{\eta g}{l}\left\{-0.5532\,P_2 + 6.558\,R_2 - 0.1740\right\}$$

$$S_2 + 0.3250 = \frac{\eta g}{l}\left\{-1.3\,P_2 - 33.038\,R_2 - 0.2014\right\}$$

$$0.5000 = \frac{\eta g}{l}\left\{-2\,P_2 - 88.08\,R_2 + 0.06365\right\}$$

$$0.3617 = \frac{\eta g}{l}\left\{-1.4468\,P_2 - 138.68\,R_2 + 0.1372\right\}$$

$$0.1867 = \frac{\eta g}{l}\left\{-0.7468\,P_2 - 37.596\,R_2 - 0.0274\right\}$$



λ = ratio of WAVE LENGTHS
σ/E = f(L/ρ)
λ = 1
λ = 0.1
λ = 0.01
K
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100
200
300
400
500
600
700
800
900
1000
1100
√(12(1-μ²)) (R/t)

Donnel's Equation (Cong. of Applied Mech.) 7

$$\frac{Et^2}{12(1-\mu^2)}\left\{r^2\nabla^8 w + \frac{2}{r^6}\frac{\partial^6 w}{\partial\theta^6} + \frac{1}{r^6}\frac{\partial^4 w}{\partial\theta^4}\right\} + E\frac{\partial^4 w}{\partial x^4} = -\sigma\frac{\partial^2}{\partial x^2}\left[r^2\nabla^4 w - \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right]$$

Put $w = \sin n\theta \sin\frac{2\pi x}{l}$

Wave length in axial direction $= l$, in circumferential direction $= \frac{2\pi}{n} r = m$, $\frac{2\pi}{m} = \frac{n}{r}$

$$\frac{t^2}{12(1-\mu^2)}\left[r^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{n}{r}\right)^2\right\}^4 - 2\left(\frac{n}{r}\right)^6 + \frac{1}{r^2}\left(\frac{n}{r}\right)^4\right] + \left(\frac{2\pi}{l}\right)^4 = \frac{\sigma}{E}\left[r^2\left(\frac{2\pi}{l}\right)^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{n}{r}\right)^2\right\}^2 + \frac{n^2}{l^2}\right]$$

or
$$\frac{t^2}{12(1-\mu^2)}\left[r^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{2\pi}{m}\right)^2\right\}^4 - 2\left(\frac{2\pi}{m}\right)^6 + \frac{1}{r^2}\left(\frac{2\pi}{m}\right)^4\right] + \left(\frac{2\pi}{l}\right)^4$$

$$= \frac{\sigma}{E}\left[r^2\left(\frac{2\pi}{l}\right)^2\left\{\left(\frac{2\pi}{l}\right)^2+\left(\frac{2\pi}{m}\right)^2\right\}^2 + \left(\frac{2\pi}{l}\right)^2\left(\frac{2\pi}{m}\right)^2\right]$$

Put $r^2 = R^2\frac{t^2}{12(1-\mu^2)}$ $\quad\left(\frac{2\pi}{l}\right)^2 = \frac{\alpha}{\frac{t^2}{12(1-\mu^2)}}$

$$\left(\frac{2\pi}{m}\right)^2 = \frac{\beta}{\frac{t^2}{12(1-\mu^2)}}$$

$$R^2(\alpha+\beta)^4 - 2\beta^3 + \frac{\beta^2}{R^2} + \alpha^2 = \frac{\sigma}{E}\left[R^2\alpha(\alpha+\beta)^2 + \alpha\beta\right]$$

or
$$\frac{\sigma}{E} = \frac{R^2(\alpha+\beta)^4 - 2\beta^3 + \frac{\beta^2}{R^2} + \alpha^2}{R^2\alpha(\alpha+\beta)^2 + \alpha\beta}$$

Let $\quad \left(\frac{m}{l}\right)^2 = \lambda = \frac{\alpha}{\beta} \quad$ or $\quad \alpha = \lambda\beta$. 2)

$$\frac{\sigma}{E} = \frac{R^2\beta^2(1+\lambda)^4 - 2\beta + \frac{1}{R^2} + \lambda^2}{R^2\lambda\beta(1+\lambda)^2 + \lambda}$$

$$\left\{R^2\lambda\beta(1+\lambda)^2+\lambda\right\}\left\{2R^2(1+\lambda)^4\beta - 2\right\}$$

$$-\left\{R^2\beta^2(1+\lambda)^4 - 2\beta + \frac{1}{R^2} + \lambda^2\right\}R^2\lambda(1+\lambda)^2 = 0$$

$$R^4\lambda(1+\lambda)^6\beta^2 + 2R^2\lambda(1+\lambda)^4\beta - \left\{\lambda(1+\lambda)^2 + 2\lambda + R^2\lambda^3(1+\lambda)^2\right\} = 0.$$

$$\lambda \sim 1, \qquad R \gg 1,$$

$$R^2(1+\lambda)^4\beta^2 + 2(1+\lambda)^2\beta - \lambda^2 = 0.$$

$$\beta^2 + \frac{2}{R^2(1+\lambda)^2}\beta - \frac{\lambda^2}{R^2(1+\lambda)^4} = 0$$

$$\beta = -\frac{1}{R^2(1+\lambda)^2} + \sqrt{\frac{1}{R^4(1+\lambda)^4} + \frac{\lambda^2}{R^2(1+\lambda)^4}}$$

$$= \frac{1}{R^2(1+\lambda)^2}\left[\sqrt{1+\lambda^2R^2} - 1\right]$$

$$\cong \frac{\lambda}{R(1+\lambda)^2}$$

$$\frac{\sigma}{E} = \frac{2\lambda^2 + \frac{1}{R^2} - 2\left[(1+\lambda)^2 + 1\right]\beta}{R^2\lambda(1+\lambda)^2\beta + \lambda}$$

$$= \frac{2\lambda^2 + \frac{1}{R^2} - \frac{2\lambda\left[(1+\lambda)^2+1\right]}{R(1+\lambda)^2}}{R\lambda^2 + \lambda} \cong \frac{2}{R}$$

$$\frac{\sigma}{E} = \frac{2}{\left(\frac{r}{t}\right)\sqrt{12(1-\mu^2)}}$$

$$\sigma = \frac{E}{\sqrt{3(1-\mu^2)}}\left(\frac{t}{R}\right)$$

Nothing new.

$$R^4(1+\lambda)^6\beta^2+2R^2(1+\lambda)^4\beta-\left\{(3+2\lambda+\lambda^2)+R^2\lambda^2(1+\lambda)^2\right\}=0 \qquad 4)$$

$$\beta^2+\frac{2}{R^2(1+\lambda)^2}\beta-\left\{\frac{(3+2\lambda+\lambda^2)}{R^4(1+\lambda)^6}+\frac{\lambda^2}{R^2(1+\lambda)^4}\right\}=0$$

$$\beta=-\frac{1}{R^2(1+\lambda)^2}+\sqrt{\frac{1}{R^4(1+\lambda)^4}+\frac{\lambda^2}{R^2(1+\lambda)^4}+\frac{3+2\lambda+\lambda^2}{R^4(1+\lambda)^6}}$$

$$=\frac{1}{R^2(1+\lambda)^2}\left[\sqrt{1+\lambda^2R^2+\frac{3+2\lambda+\lambda^2}{(1+\lambda)^2}}-1\right]$$

$$\frac{\sigma}{E}=\frac{R^2(1+\lambda)^4\beta^2-2\beta+\frac{1}{R^2}+\lambda^2}{R^2\lambda(1+\lambda)^2\beta+\lambda}$$

$$=\frac{\frac{(3+2\lambda+\lambda^2)}{R^2(1+\lambda)^2}+2\lambda^2-2\left[(1+\lambda)^2+1\right]\beta+\frac{1}{R^2}}{R^2\lambda(1+\lambda)^2\beta+\lambda}$$

$$\boxed{\frac{\sigma}{E}=\frac{\frac{(4+4\lambda+2\lambda^2)}{R^2(1+\lambda)^2}+2\lambda^2-2(2+2\lambda+\lambda^2)\beta}{\lambda\left[R^2(1+\lambda)^2\beta+1\right]}}$$

$$\frac{\sigma}{E} = \frac{\dfrac{(4+4\lambda+2\lambda^2)}{R^2(1+\lambda)^2} + 2\lambda^2 - 2(2+2\lambda+\lambda^2)\beta}{\lambda\sqrt{1+\lambda^2R^2+\dfrac{3+2\lambda+\lambda^2}{(1+\lambda)^2}}}$$

for $\lambda = 1$

$$\frac{\sigma}{E} = \frac{\frac{9}{4}\frac{1}{R^2} + 2 - 10\left\{\dfrac{\sqrt{1+R^2+\frac{5}{4}} - 1}{4R^2}\right\}}{\sqrt{1+R^2+\frac{5}{4}}}$$

$$\frac{\sigma}{E} = \frac{\frac{1}{4R^2}\left[19 - 10\sqrt{1+R^2+\frac{5}{4}}\right] + 2}{\sqrt{1+R^2+\frac{5}{4}}} = \frac{\frac{19}{4R^2}+2}{\sqrt{1+R^2+\frac{5}{4}}} - \frac{5}{2R^2}$$

$$\boxed{\frac{\sigma}{E} = \frac{2+\frac{19}{4R^2}}{\sqrt{R^2+\frac{9}{4}}} - \frac{5}{2R^2}}$$

$\lambda = 100$

$$\frac{\sigma}{E} = \frac{\dfrac{10403}{R^2\,10201} + 20000 - 2\times 10202\,\dfrac{\sqrt{1+10000R^2+\frac{10202}{10201}} - 1}{R^2\,10201}}{100\sqrt{1+10000R^2+\frac{10203}{10201}}}$$

$$\simeq \frac{\frac{1.020}{R^2} + 20000 - \frac{200}{R}}{10000R} = \frac{2}{R} - \frac{0.02}{R^2} + \frac{0.000102}{R^3}$$

$$= \frac{2}{R}\left(1 - \frac{0.01}{R} + \frac{0.000051}{R^2}\right)$$

$\lambda = 0.1$

$$\frac{\sigma}{E} = \frac{\frac{4.42}{1.21}\frac{1}{R^2} + 0.02 - 4.42\frac{\sqrt{1+0.01R^2+\frac{2.21}{1.21}} - 1}{1.21\,R^2}}{0.1\sqrt{1+0.01R^2+\frac{2.21}{1.21}}}$$

$$= \frac{\frac{3.6540}{R^2} + 0.02 - \frac{3.6529}{R^2}\left\{\sqrt{3.652 + 0.01R^2} - 1\right\}}{0.1\sqrt{2.8264 + 0.01R^2}}$$

If $R = 100$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.036540 + 2 - 0.036529 \times 10.80}{0.1 \times 11.80}\right]$$ 0.422

$R = 300$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.01218 + 6 - 0.012176 \times 29.048}{0.1 \times 30.048}\right]$$ $K = 0.390$

$R = 1000$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.0028264 + 20 - 0.0036529 \times 100}{0.1 \times 101}\right] =$$ .595

$R = 10.$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.365401 + 0.2 - 0.36529 \times \overset{1.164}{\cancel{2.956}}}{0.1 \times 2.964}\right]$$ .214

b)

$\lambda = 0.01$

$$\frac{\sigma}{E} = \frac{\frac{2.0402}{1.0201}\frac{1}{R^2} + 0.0002 - 4.0402\left\{\frac{\sqrt{1 + \frac{3.0201}{1.0201} + 0.0001R^2} - 1}{1.0201R^2}\right\}}{0.01\sqrt{1 + \frac{2.0201}{1.0201} + 0.0001R^2}}$$

for. $R = 100$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.036540 + 0.02 - \frac{4.0402}{1.0201}\,0.612261}{0.01 \times 2.2351}\right]$$

3.960

$$= \frac{1}{R}\left[\frac{0.010391}{0.01 \times 1.9951}\right] = \frac{2}{R} \times 0.2604. \qquad k = 0.7578$$

1.8

$R = 300$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.01218 + 0.06 - \frac{4.0402}{1.0201} \times \frac{2.6613}{300}}{0.01 \times 3.6060}\right]$$

3.9

$$= \frac{1}{R}\;\frac{0.11636}{0.0163613}$$

0.3275

$R = 1000$

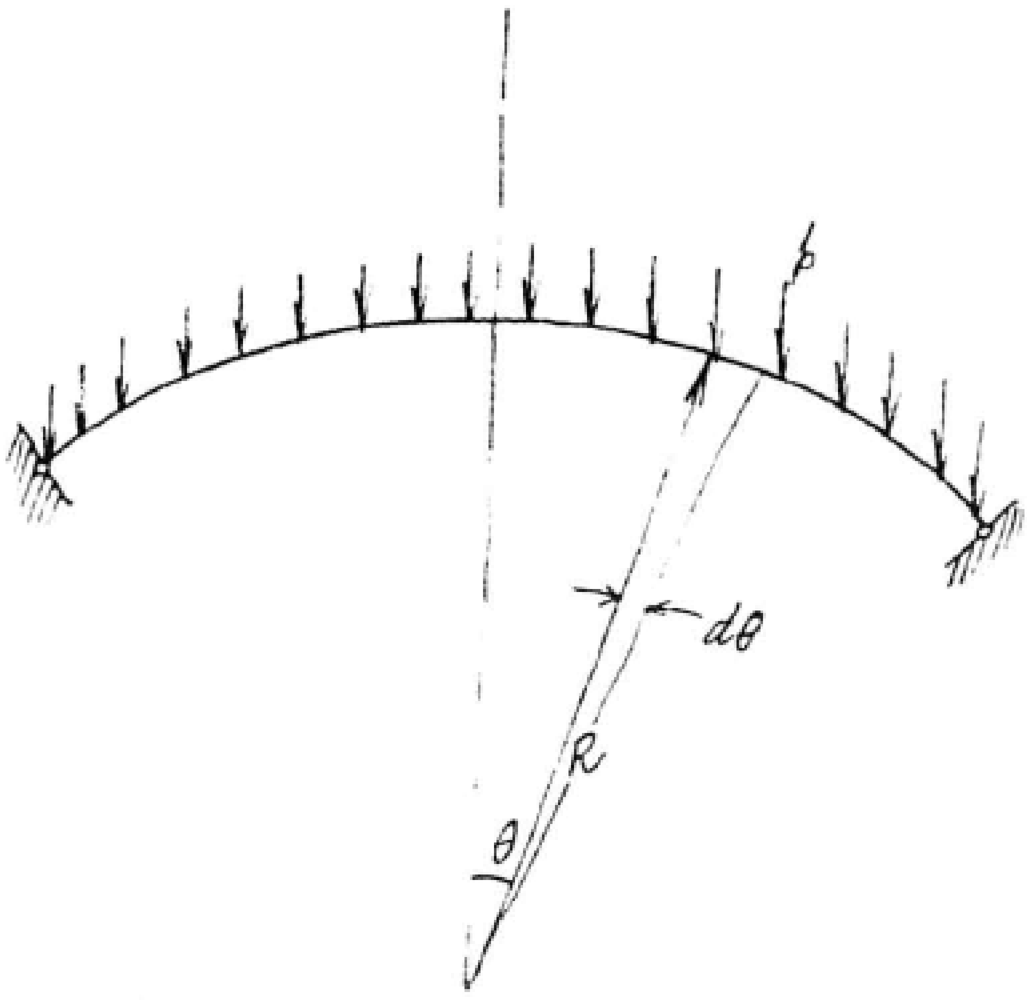
$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.0029803 + 0.2 - 3.960 \times 0.01049}{0.01 \times 11.49}\right] = \qquad 0.426.$$

$R = 50$

$$\frac{\sigma}{E} = \frac{1}{R}\left[\frac{0.07308 + 0.01 - 3.960 \times \frac{1.052}{50}}{0.01 \times 2.052}\right] \qquad \sim 0$$

Section 4

Buckling of Spherical Shell



The original form of the shell is spherical.

Now suppose the deflected form of the shell is axially symmetrical.

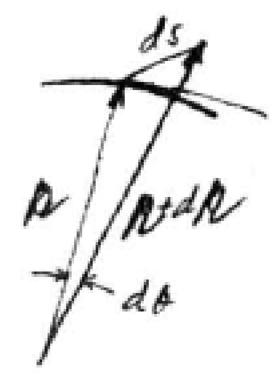
$$\theta_1 = \theta_0 + \theta_0 f(\theta_0) = \theta_0[1+f(\theta_0)]$$

$$R = R_0 + R\, g(\theta_0)$$

$$= R[1+g(\theta_0)]$$

The original length of the element $(ds)_0 = R(d\theta)_0$

The new length of the element



$$= \sqrt{R^2(d\theta_1)^2 + (dR_1)^2}$$

$$= \sqrt{R^2[1+g(\theta_0)]^2\left[\{1+f(\theta_0)\}d\theta_0 + \theta_0 f'(\theta_0)\, d\theta_0\right]^2 + R^2[g'(\theta_0)]^2(d\theta_0)^2}$$

$$= R\sqrt{[1+g(\theta_0)]^2[1+f(\theta_0)+\theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2}\; d\theta_0$$

If the deflection is inextensional in the sense that $(ds)_0 = (ds)_1$

Then $$\underline{[1+g(\theta_0)]^2[1+f(\theta_0)+\theta_0 f'(\theta_0)]^2+[g'(\theta_0)]^2 = 1}$$

2)

The distance of the element from the axis is

$\left(\frac{1}{2} E \varepsilon^2 v\right)$

$$R \cdot \sin\theta_0$$

before deflection.

The distance is $R \sin\theta_1$ after deflection.

$$R[1+g(\theta_0)] \sin[\theta_0(1+f(\theta_0))]$$

$\sin[\theta_0 + \theta_0 f(\theta_0)]$

The change in length of the ring ds is

$$2\pi R\left[[1+g(\theta_0)] \sin\{\theta_0(1+f(\theta_0))\} - \sin\theta_0\right]$$

The strain energy stored in this ds is

$$\frac{1}{2}[E\,\varepsilon]\,t\,ds \cdot 2\pi R$$

$t =$ thickness

Now $$\varepsilon = [1+g(\theta_0)] \frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1$$

$$= \cancel{[1+g(\theta_0)]\{\cos\{\theta_0 f(\theta_0)\} + \cot\theta \sin}$$

The total strain energy

$$= \frac{1}{2} t E R^2 2\pi \int_0^{\alpha} \left\{[1+g(\theta_0)] \frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1\right\}^2 \sin\theta_0 \, d\theta_0$$

Potential energy of the pressure force.

pV where $V =$ volume under the <u>shell</u>

The volume under the shell

$$= \int_0^{\alpha} \frac{1}{3}\, 2\pi r \sin\theta_1 \, \cancel{d\theta} \cdot r \cdot r \, d\theta_1$$

$$= \frac{2\pi R^3}{3} \int_0^{\alpha} [1+g(\theta_0)]^3 \sin\{\theta_0[1+f(\theta_0)]\} \{[1+f(\theta_0)] + \theta_0 f'(\theta_0)\} \, d\theta_0$$

The integral to be minimized is

$$\left\{\frac{t}{R}E\right\} \int_0^{\alpha} \left\{[1+g(\theta_0)] \frac{\sin\{\theta_0[1+f(\theta_0)]\}}{\sin\theta_0} - 1\right\}^2 \sin\theta_0 \, d\theta_0$$

$$- \frac{2p}{3} \int_0^{\alpha} [1+g(\theta_0)]^3 \sin\{\theta_0[1+f(\theta_0)]\} \times [1+f(\theta_0) + \theta_0 f'(\theta_0)] \, d\theta_0 .$$

To simplify the expression, let us put $\underline{\theta_0 f(\theta_0)} = h(\theta_0)$

$$\boxed{\begin{aligned} I = \left\{\frac{t}{R}E\right\} &\int_0^{\alpha} \left\{[1+g(\theta_0)] \frac{\sin(\theta_0 + h(\theta_0))}{\sin\theta_0} - 1\right\}^2 \sin\theta_0 \, d\theta_0 \\ - \frac{2p}{3} &\int_0^{\alpha} [1+g(\theta_0)]^3 \sin(\theta_0 + h(\theta_0)) \cdot [1+h'(\theta_0)] \, d\theta_0 \end{aligned}}$$

The inextensional condition is

$$[1+g(\theta_0)]^2 [1+h'(\theta_0)]^2 + [g'(\theta_0)]^2 - 1 = 0$$

~~[illegible]~~ 4)

The Euler-Lagrange differential equation is then

$$\frac{t}{R} E \left\{ \cancel{2} \left[[1+g(\theta_0)] \frac{\sin(\theta_0 + h(\theta_0))}{\sin\theta_0} - 1 \right] \cancel{\sin\theta_0} \cdot \frac{\sin(\theta_0 + h(\theta_0))}{\cancel{\sin\theta_0}} \right\}$$

$$- \frac{\cancel{2}p}{\cancel{3}} \left\{ \cancel{3} \left[1 + g(\theta_0)\right]^2 \sin(\theta_0 + h(\theta_0)) \cdot [1 + h'(\theta_0)] \right\}$$

$$+ \lambda \left\{ \cancel{2} [1 + g(\theta_0)][1 + h'(\theta_0)]^2 \right\} - \cancel{2} g''(\theta_0) \bigg\} \overset{\frac{d}{d\theta_0}(\lambda g'(\theta_0))}{} = 0.$$

$$\frac{t}{R} E \left\{ \cancel{2} \sin\theta_0 \left[[1+g(\theta_0)] \frac{\sin(\theta_0 + h(\theta_0))}{\sin\theta_0} - 1 \right] [1 + g(\theta_0)] \frac{\cos(\theta_0 + h(\theta_0))}{\sin\theta_0} \right\}$$

$$- \frac{\cancel{2}p}{3} \left\{ \cancel{[1 + g(\theta_0)]^3 \cos(\theta_0 + h(\theta_0)) [1 + h'(\theta_0)]} \right.$$

$$\left. - 3 [1 + g(\theta_0)]^2 g'(\theta_0) \sin(\theta_0 + h(\theta_0)) - \cancel{[1 + g(\theta_0)]^3 \cos(\theta_0 + h(\theta_0)) [1 + h'(\theta_0)]} \right\}$$

$$\frac{d}{d\theta_0} \lambda \left\{ \frac{d}{d\theta_0} \left[[1 + g(\theta_0)]^2 \cancel{2} [1 + h'(\theta_0)] \right\} \right\} = 0.$$

5)

$$\frac{t}{R}E\left\{[1+g(\theta_0)]\frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0}-1\right\}\sin(\theta_0+h(\theta_0))$$

$$-\ p\,[1+g(\theta_0)]^2[1+h'(\theta_0)]\sin(\theta_0+h(\theta_0))$$

$$+\lambda\left\{[1+g(\theta_0)][1+h'(\theta_0)]^2-g''(\theta_0)\right\}-g'(\theta_0)\frac{d\lambda}{d\theta_0}=0. \qquad (1)$$

$$\frac{t}{R}E\left\{[1+g(\theta_0)]\frac{\sin(\theta_0+h(\theta_0))}{\sin\theta_0}-1\right\}[1+g(\theta_0)]\cos(\theta_0+h(\theta_0))$$

$$-\frac{p}{3}\Big\{\cancel{[1+g(\theta_0)]^3\cos(\theta_0+h(\theta_0))\,[1+h'(\theta_0)]}$$

$$-3[1+g(\theta_0)]^2g'(\theta_0)\sin(\theta_0+h(\theta_0))-\cancel{[1+g(\theta_0)]^3[1+h'(\theta_0)]\cos(\theta_0+h(\theta_0))}\Big\}$$

$$-\lambda\left\{2[1+g(\theta_0)]g'(\theta_0)[1+h'(\theta_0)]+[1+g(\theta_0)]^2h''(\theta_0)\right\}\cancel{=0} \qquad (2)$$

$$-[1+g(\theta_0)]^2[1+h'(\theta_0)]\frac{d\lambda}{d\theta_0}=0$$

$$[1+g(\theta_0)]^2[1+h'(\theta_0)]^2+[g'(\theta_0)]^2-1=0 \qquad (3)$$

Now if we assume that both

$$g(\theta_0) \text{ and } h(\theta_0)$$

are small quantities ~~and so the quadratic & higher order~~

~~terms~~ But retain terms of the form $g''(\theta_0)\cdot g(\theta_0)$ $\quad g''(\theta_0)\, g'(\theta_0)$

Then $$\frac{\sin[\theta_0 + h(\theta_0)]}{\sin\theta_0} = 1 + \cot\theta_0 \cdot h(\theta_0)$$

[In the following calculation $\theta_0 \approx \theta$]

$$[1+g(\theta)]\frac{\sin[\theta+h(\theta)]}{\sin\theta} - 1$$

$$= [1+g(\theta)][1+h(\theta)\cot\theta] - 1$$

$$\cong g(\theta) + h(\theta)\cot\theta$$

$$\sin(\theta + h(\theta)) = \sin\theta + h(\theta)\cos\theta$$

$$[1+g(\theta)]^2[1+h'(\theta)][\sin\theta + h(\theta)\cos\theta]$$

$$= [1+2g(\theta)][1+h'(\theta)][\sin\theta + h(\theta)\cos\theta]$$

$$= [1+2g(\theta)+h'(\theta)][\sin\theta + h(\theta)\cos\theta)]$$

$$= \sin\theta + h(\theta)\cos\theta + 2g(\theta)\cdot\sin\theta + h'(\theta)\sin\theta$$

$$[1+g(\theta)][1+2h'(\theta)] - g''(\theta_0)$$

$$= 1 + g(\theta) + 2h'(\theta) - g''(\theta)$$

(7)

Then the first differential equation becomes

$$\frac{t}{R}E\left\{\sin\theta\left[g'(\theta)+h(\theta)\cot\theta\right]\right\}_{\theta} - p\left\{\sin\theta + h(\theta)\cos\theta\right.$$

$$\left. + 2g(\theta)\sin\theta + h'(\theta)\sin\theta\right\}$$

$$+\lambda\left\{1+g(\theta)+2h'(\theta)-g''(\theta)\right\} - g'(\theta)\frac{d\lambda}{d\theta} = 0.$$

$$\cancel{\frac{t}{R}E\sin\theta}$$

$$\cancel{-p\sin\theta + \left[\frac{t}{R}E\sin\theta - 2p\sin\theta\right]}$$

$$\cos(\theta_0 + h(\theta)) = \cos\theta - h(\theta)\sin\theta$$

$$[1+g'(\theta)]\cos(\theta+h(\theta)) = \cos\theta - h(\theta)\sin\theta + g(\theta)\cos\theta.$$

$$\frac{t}{R}E\left\{\cos\theta\left[g(\theta)+h(\theta)\cot\theta\right]\right\} + p\sin\theta\cdot g'(\theta)$$

$$\cancel{-\frac{p}{3}\left\{[1+3g(\theta)]\cos\theta - h(\theta)\sin\theta + h'(\theta)\cos\theta\right\}}$$

$$\cancel{-3\left[\sin\theta\cdot g(\theta)\right]}$$

$$-\lambda\left\{2g'(\theta)+h''(\theta)\right\} - \left\{1+2g(\theta)+h'(\theta)\right\}\frac{d\lambda}{d\theta} = 0$$

$$[1+2g(\theta)][1+2h'(\theta)] - 1 = 0$$

$$\underline{\underline{g(\theta) + h'(\theta) = 0.}}$$

From the last equation, it is seen that $g' \simeq h''$ 8)
there $h'''\cdot h$, $h'''\cdot h''$, $h'''\cdot h'$ not to be neglected.

$$\frac{t}{R}E\left[\, h(\theta)\cos\theta - h'(\theta)\sin\theta \,\right] - p\left\{ \sin\theta + h(\theta)\cdot\cos\theta - h'(\theta)\sin\theta \right\}$$

$$\cancel{-2h(\theta)\sin\theta\,\lambda} \quad + \lambda\left\{ 1 + h'(\theta) + h'''(\theta) \right\} + h''(\theta)\frac{d\lambda}{d\theta} = 0.$$

$$\frac{t}{R}E\left[\, h(\theta)\cdot\cos\theta\cdot\cot\theta - h'(\theta)\cos\theta \,\right] - p\, g''(\theta)\sin\theta$$

$$+ \ \lambda\left[\, h''(\theta) \,\right] - \left\{ 1 - h'(\theta) \right\}\frac{d\lambda}{d\theta} = 0.$$

This method of derivation unsatisfactory, because we have to use further differentiation to eliminate λ but then some of the neglected terms may become important.

Timoshenko's differential equations when there is no bending moment will be 9)

$$\frac{dN_x}{d\theta} + (N_x - N_y)\cot\theta + N_y\left(\frac{x}{a} + \frac{dw}{a\,d\theta}\right) = 0$$

$$N_x + N_y + qa + N_x\left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right) + N_y\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\cot\theta = 0$$

$$N_x = \frac{Et}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2) - \frac{qa}{2}$$

$$= \frac{Et}{1-\nu^2}\left[\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right) + \nu\left(\frac{u\cos\theta}{a\sin\theta} - \frac{w}{a}\right)\right] - \frac{qa}{2}$$

$$N_y = \frac{Et}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1) - \frac{qa}{2}$$

$$= \frac{Et}{1-\nu^2}\left[\left(\frac{u\cos\theta}{a\sin\theta} - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] - \frac{qa}{2}$$

$$\frac{dN_x}{d\theta} = \frac{Et}{1-\nu^2}\left[\frac{1}{a}\frac{d^2u}{d\theta^2} - \frac{1}{a}\frac{dw}{d\theta} + \nu\left(\frac{1}{a}\frac{\cos\theta}{\sin\theta}\frac{du}{d\theta} - \frac{u}{a}\frac{1}{\sin^2\theta} - \frac{1}{a}\frac{dw}{d\theta}\right)\right]$$

Therefore the differential equation can be written as

$$\frac{Et}{1-\nu^2}\left[\frac{1}{a}\frac{d^2u}{d\theta^2} - (1+\nu)\frac{1}{a}\frac{dw}{d\theta} + \nu\left(\frac{1}{a}\cot\theta\frac{du}{d\theta} - \frac{u}{a}\frac{1}{\sin^2\theta}\right)\right]$$

$$+ \frac{Et}{1-\nu^2}\left[\left(\frac{du}{a\,d\theta} - \frac{u\cos\theta}{a\sin\theta}\right)(1-\nu)\right]\cot\theta$$

$$+ \left\{\frac{Et}{1-\nu^2}\left[\left(\frac{u\cos\theta}{a\sin\theta} - \frac{u}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] - \frac{qa}{2}\right\}\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0.$$

10)

Let us put ~~ghr~~ $p = \dfrac{q}{\left(\frac{Et}{1-\nu^2}\right)}$, then

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - (1+\nu)\frac{dw}{a\,d\theta} + \nu\left(\frac{1}{a}\cot\theta\frac{du}{d\theta} - \frac{u}{a}\frac{1}{\sin^2\theta}\right)$$

$$+ (1-\nu)\cot\theta\left(\frac{du}{a\,d\theta} - \frac{u}{a}\cot\theta\right)$$

$$+ \left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\left[\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right) - \frac{pa}{2}\right] = 0.$$

$$\cancel{\frac{1}{a}\frac{d^2u}{d\theta^2}}$$

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - (1+\nu)\frac{dw}{a\,d\theta} + \cot\theta\frac{du}{a\,d\theta} - \frac{u}{a}\cot^2\theta - \nu\frac{u}{a}$$

$$- \frac{pa}{2}\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) + \left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\left[\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] = 0$$

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - \left[1+\nu+\frac{pa}{2}\right]\frac{dw}{a\,d\theta} + \cot\theta\frac{du}{a\,d\theta} - \left(\cot^2\theta + \nu + \frac{pa}{2}\right)\frac{u}{a}$$

$$+ \left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\left[\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right) + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] = 0.$$

$$\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta - \frac{2w}{a}\right)(1+\nu) + \cancel{pa}$$

$$+ \left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right)\left[\frac{du}{a\,d\theta} - \frac{w}{a} + \nu\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right) - \frac{pa}{2}\right]$$

$$+ \cot\theta\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\left[\frac{u}{a}\cot\theta - \frac{w}{a} + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right) - \frac{pa}{2}\right] = 0.$$

(19)

$$\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta - \frac{2w}{a}\right)(1+\nu) - \frac{pa}{2}\left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta} + \cot\theta\,\frac{u}{a} + \cot\theta\,\frac{dw}{a\,d\theta}\right)$$

$$\cancel{\left(1+\nu-\frac{pa}{2}\right)\frac{du}{a\,d\theta} + \left(1+\nu-\frac{pa}{2}\right)\frac{u}{a}\cot\theta - \frac{pa}{2}\frac{d^2w}{a\,d\theta^2} - \frac{pa}{2}\cot\theta\,\frac{dw}{a\,d\theta} - (1+\nu)\frac{2w}{a}}$$

$$\left(1+\nu-\frac{pa}{2}\right)\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta\right) - \frac{pa}{2}\frac{d^2w}{a\,d\theta^2} - \frac{pa}{2}\cot\theta\,\frac{dw}{a\,d\theta} - (1+\nu)\frac{2w}{a}$$

$$+\left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right)\left[\frac{du}{a\,d\theta} - \frac{w}{a} + \nu\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right)\right]$$

$$+\cot\theta\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right)\left[\frac{u}{a}\cot\theta - \frac{w}{a} + \nu\left(\frac{du}{a\,d\theta} - \frac{w}{a}\right)\right] = 0.$$

Neglect the change in curvature in y-~~ax~~ direction, we have

$$\cancel{\frac{1}{a}\frac{d^2u}{d\theta^2} - \left[1+\nu+\frac{pa}{2}\right]\frac{dw}{a\,d\theta} + \cot\theta\,\frac{du}{a\,d\theta} - \left(\cot^2\theta - \nu\,\frac{u}{a}\right)}$$

$$\frac{1}{a}\frac{d^2u}{d\theta^2} - \left[1+\nu+\frac{pa}{2}\right]\frac{dw}{a\,d\theta} + \cot\theta\,\frac{du}{a\,d\theta} - \left(\cot^2\theta + \nu + \frac{pa}{2}\right)\frac{u}{a} = 0.$$

$$\left(1+\nu-\frac{pa}{2}\right)\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta\right) - \frac{pa}{2}\left(\frac{d^2w}{a\,d\theta^2} + \cot\theta\,\frac{dw}{a\,d\theta}\right) - (1+\nu)\frac{2w}{a}$$

$$+\left(\frac{d^2w}{a\,d\theta^2} + \frac{du}{a\,d\theta}\right)\left[\frac{du}{a\,d\theta} - \frac{w}{a} + \nu\left(\frac{u}{a}\cot\theta - \frac{w}{a}\right)\right] = 0.$$

Putting $u = \frac{d\psi}{d\theta}$, we have from the first equation by integrating 10)

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - (1+\nu)(\psi+w) - \frac{pa}{2}(\psi+w) = 0$$

Similarly, by putting $u = \frac{d\psi}{d\theta}$ into the second equation, we have

$$(1+\nu-\frac{pa}{2})\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \frac{pa}{2}\left(\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta}\right)$$
$$-(1+\nu)\,2w + \left(\frac{d^2w}{a\,d\theta^2} + \frac{d^2\psi}{a\,d\theta^2}\right)\left(\frac{d^2\psi}{d\theta^2} - \frac{w}{\cancel{a}}(1+\nu) + \nu\cot\theta\frac{d\psi}{d\theta}\right) = 0.$$

Let $\frac{u}{a} \sim (u)$, $\frac{w}{a} \sim (w)$, $\frac{pa}{2} = \phi$.

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + 2\psi - (1+\nu)(\psi+w) - \phi(\psi+w) = 0$$

$$(1+\nu-\phi)\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \phi\left(\frac{d^2w}{d\theta^2} + \cot\theta\frac{dw}{d\theta}\right) - 2(1+\nu)w$$
$$+\left(\frac{d^2w}{d\theta^2} + \frac{d^2\psi}{d\theta^2}\right)\left(\frac{d^2\psi}{d\theta^2} + \nu\cot\theta\frac{d\psi}{d\theta} - (1+\nu)w\right) = 0.$$

or $$(1+\nu)\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta}\right) - \phi\left(\frac{d^2(\psi+w)}{d\theta^2} + \cot\theta\frac{d(\psi+w)}{d\theta}\right) - 2(1+\nu)w$$
$$+\left[\frac{d^2(\psi+w)}{d\theta^2}\right]\left[\frac{d^2\psi}{d\theta^2} + \nu\cot\theta\frac{d\psi}{d\theta} - (1+\nu)w\right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - [(1+\nu)+\phi]\gamma = 0$$

$$(1+\nu)[-1+\nu+\phi]\gamma - \phi\left[\frac{d^2\gamma}{d\theta^2} + \cot\theta\frac{d\gamma}{d\theta}\right] - \cancel{2(1+\nu)\gamma}$$

$$+ \frac{d^2\gamma}{d\theta^2}\left[\frac{d^2\psi}{d\theta^2} + \nu\cot\theta\frac{d\psi}{d\theta} + (1+\nu)\psi - (1+\nu)\gamma\right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + 2\psi - [1+\nu+\phi]\gamma = 0$$

$$\frac{d^2\gamma}{d\theta^2}\left[-(1-\nu)\cot\theta\frac{d\psi}{d\theta} - (1-\nu)\psi + \phi\gamma\right] + (1+\nu)[\phi-(1-\nu)]\gamma$$

$$- \phi\left[\frac{d^2\gamma}{d\theta^2} + \cot\theta\frac{d\gamma}{d\theta}\right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta\frac{d\psi}{d\theta} + 2\psi - [1+\nu+\phi]\gamma = 0$$

$$(1+\nu)[\phi-(1-\nu)]\gamma - \frac{d^2\gamma}{d\theta^2}\left[(1-\nu)\cot\frac{d\psi}{d\theta} + (1-\nu)\psi - \phi\gamma\right] - \phi\left[\frac{d^2\gamma}{d\theta^2} + \cot\theta\frac{d\gamma}{d\theta}\right] = 0.$$

Neglecting the curvature term, we have 14)

$$H(\psi) - (1+\nu)(\psi+w) - \phi(\psi+w) = 0.$$

$$(1+\nu-\phi)\left(\frac{d^2\psi}{d\theta^2} + \cot\theta\,\frac{d\psi}{d\theta}\right) - \phi\left(\frac{d^2w}{d\theta^2} + \cot\theta\,\frac{dw}{d\theta}\right) - (1+\nu)$$

$$= 0$$

$$(1+\nu-\phi)\left[\,H(\psi) - 2\psi\right] - \phi\left(\,H(w) - 2w\right) - (1+\nu)\,\underline{2w}$$

Put in $\quad \psi = \sum_{n=0}^{\infty} A_n P_n$

$$w = \sum_{n=0}^{\infty} B_n P_n$$

$$\sum_{n=0}^{\infty}\left[-A_n \lambda_n - (1+\nu+\phi)(A_n + B_n)\right] P_n = 0$$

$$\sum_{n=0}^{\infty}\Big[(1+\nu-\phi)\left[-\lambda_n A_n - 2A_n\right] - \phi\left[-\lambda_n B_n - 2B_n\right]$$

$$-2(1+\nu)B_n \Big] P_n = 0.$$

$$\sum_{n=0}^{\infty}\left[\,(1+\nu+\phi+\lambda_n)A_n + (1+\nu+\phi)B_n\right] P_n = 0$$

$$\sum_{n=0}^{\infty}\left[\,(1+\nu-\phi)(2+\lambda_n)A_n - \left\{\phi(2+\lambda_n) - 2(1+\nu)\right\}B_n\right] P_n = 0.$$

15)

The set of homogeneous equation for A_n and B_n is

$$(1+\nu+\phi+\lambda_n)A_n + (1+\nu+\phi)B_n = 0$$

$$(1+\nu-\phi)(2+\lambda_n)A_n + \{2(1+\nu)-\phi(2+\lambda_n)\}B_n = 0.$$

The determinant must be zero, so

$$(1+\nu+\phi+\lambda_n)\{2(1+\nu)-\phi(2+\lambda_n)\}$$

$$-(1+\nu+\phi)(1+\nu-\phi)(2+\lambda_n) = 0.$$

$$2(1+\nu+\phi+\lambda_n)(1+\nu) - (1+\nu+\lambda_n)(2+\lambda_n)\phi - \cancel{\phi^2(2+\lambda_n)}$$

$$-(2+\lambda_n)(1+\nu+\phi)(1+\nu) + (2+\lambda_n)(1+\nu)\phi + \cancel{\phi^2(2+\lambda_n)} = 0$$

$$\cancel{2(1+\nu)^2 + 2(1+\nu)(\phi+\lambda_n) - (2+\lambda_n)\lambda_n\phi}$$

$$\cancel{-2(1+\nu)^2 - \lambda_n(1+\nu+\phi)(1+\nu) - 2\phi(1+\nu) = 0.}$$

$$\cancel{2(1+\nu)\lambda_n - (2+\lambda_n)\lambda_n\phi - \lambda_n(1+\nu)\phi - \lambda_n(1+\nu)^2 = 0}$$

$$\cancel{2(1+\nu)\phi} + 2(1+\nu)\lambda_n$$

$$\cancel{2(1+\nu)^2} + \cancel{2(1+\nu)(\phi+\lambda_n)} - \lambda_n(2+\lambda_n)\phi$$

$$-\cancel{2(1+\nu)^2} - \cancel{2(1+\nu)\phi} - \lambda_n(1+\nu)^2 - \lambda_n\phi(1+\nu) = 0.$$

$$\cancel{\lambda_n}(1+\nu)[2-(1+\nu)] - \cancel{\lambda_n}\phi[2+\lambda_n+1+\nu] = 0.$$

$$(1+\nu)(1-\nu) - \phi[2+\lambda_n+1+\nu] = 0.$$

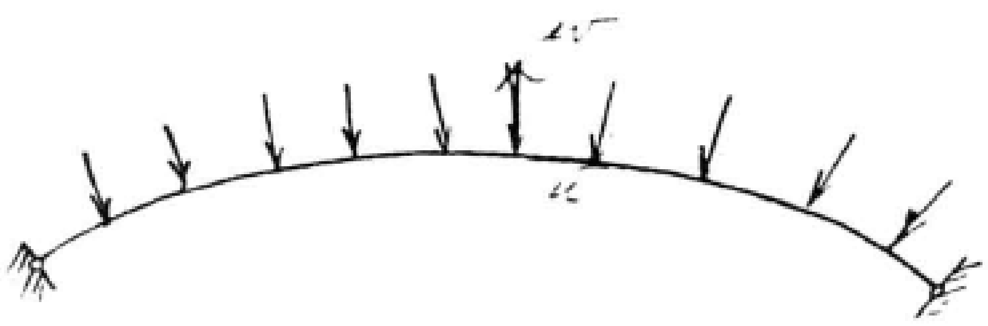
$$\phi = \frac{1-\nu^2}{3+\nu+\lambda_n}$$ Min. $\phi = 0$

$$\frac{pa}{2} = \frac{qa(1-\nu^2)}{2Et} = \frac{1-\nu^2}{3+\nu+\lambda_n}$$ 1b

$$\frac{q}{2E}\left(\frac{a}{t}\right) = \frac{1}{3+\nu+\lambda_n}$$

$$q_{cr} = \frac{2E}{3+\nu+\lambda_n}\left(\frac{t}{a}\right)$$

$$\sigma_{cr} = \frac{\cancel{2}E}{3+\nu+\lambda_n}\left(\frac{t}{\cancel{a}}\right)\left(\frac{\cancel{a}}{\cancel{t}}\right)\frac{1}{\cancel{2}} = \left(\frac{E}{3+\nu+\lambda_n}\right)$$



$$\theta_1 = \theta_0 + \frac{u}{a}$$

$$r = a + w$$

The original length of the element $(ds)_0 = a\,(d\theta_0)$

The new length of the element

$$= \sqrt{r^2 (d\theta)_1^2 + (dr_1)^2} = a \sqrt{\left(1+\frac{w}{a}\right)^2\left(1+\frac{1}{a}\frac{du}{d\theta}\right)^2 + \frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2}\; d\theta$$

~~Neglecting quadratic terms of deflections~~

$$= a\,d\theta \left\{ 1 + \frac{w}{a}\left(2+\frac{w}{a}\right) + \frac{1}{a}\frac{du}{d\theta}\left(2+\frac{1}{a}\frac{du}{d\theta}\right) + \frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2 \right\}^{\frac{1}{2}}$$

The distance of the element from the axis is

$a \sin\theta_0$ before deflection

The distance is $r \sin\theta_1$ after deflection

The change in length of the ring ds is

$$2\pi a\left[\left(1+\frac{w}{a}\right)\sin\left(\theta+\frac{u}{a}\right) - \sin\theta\right]$$

The change per unit length (~~circumferential~~ latitude)

$$= \text{at} \quad \frac{\left(1+\frac{w}{a}\right)\sin\left(\theta+\frac{u}{a}\right)}{\sin\theta} - 1$$

The change per unit length (meridian)

$$\left\{1+\frac{1}{a}\frac{du}{d\theta}\left(2+\frac{1}{a}\frac{du}{d\theta}\right)+\frac{w}{a}\left(2+\frac{w}{a}\right)+\frac{1}{a}\frac{dw}{d\theta}\left(\frac{1}{a}\frac{dw}{d\theta}\right)\right\}^{\frac{1}{2}}-1$$

$$\approx \frac{1}{a}\frac{du}{d\theta}\left(1+\frac{1}{2a}\frac{du}{d\theta}\right)+\frac{w}{a}\left(1+\frac{w}{2a}\right)+\frac{1}{a}\frac{dw}{d\theta}\left(+\frac{1}{2a}\frac{dw}{d\theta}\right)$$

$$-\frac{1}{2}\frac{1}{a^2}\left(\frac{du}{d\theta}\right)^2-\frac{1}{2}\frac{1}{a^2}w^2-\cancel{\frac{1}{2a^2}\left(\frac{dw}{d\theta}\right)^2}$$

$$\approx \underline{\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\cancel{\frac{1}{a}\frac{dw}{d\theta}}} \; +\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2$$ [up to second order terms]

$$\frac{\left(1+\frac{w}{a}\right)\sin\left(\theta+\frac{u}{a}\right)}{\sin\theta} - 1$$

$$= \left(1+\frac{w}{a}\right)\left[\cos\left(\frac{u}{a}\right)+\cot\theta\,\sin\left(\frac{u}{a}\right)\right]-1$$

$$\cong \left(1+\frac{w}{a}\right)\left[1-\frac{1}{2}\left(\frac{u}{a}\right)^2+\cot\theta\cdot\left(\frac{u}{a}\right)\right]-1$$

$$\cong -\frac{1}{2}\left(\frac{u}{a}\right)^2+\cot\theta\cdot\left(\frac{u}{a}\right)+\cot\theta\cdot\left(\frac{u}{a}\right)\left(\frac{w}{a}\right)+\frac{w}{a}$$

The stress in latitude direction

$$\frac{E}{1-\nu^2}\left[\frac{w}{a}+\frac{u}{a}\cot\theta-\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{uw}{a^2}\cot\theta+\nu\left\{\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2\right\}\right]$$

$$-\frac{pa}{2t}$$

The stress in meridian direction

$$\frac{E}{1-\nu^2}\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2+\nu\left\{\frac{w}{a}+\frac{u}{a}\cot\theta-\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{uw}{a^2}\cot\theta\right\}\right]$$

$$-\frac{pa}{2t}$$

The strain energy retaining only terms up to second order

$$2\pi a^2 t\,\frac{E}{1-\nu^2}\left\{\frac{1}{2}\int\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\nu\left\{\frac{w}{a}+\frac{u}{a}\cot\theta\right\}\right]\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right]\sin\theta\,d\theta\right.$$

$$\left.+\frac{1}{2}\int\left[\frac{w}{a}+\frac{u}{a}\cot\theta+\nu\left\{\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right\}\right]\left[\frac{w}{a}+\frac{u}{a}\cot\theta\right]\sin\theta\,d\theta\right\}$$

$$-\frac{pa}{2t}\,2\pi a^2 t\int\left[\frac{w}{a}+\frac{u}{a}\cot\theta-\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{uw}{a^2}\cot\theta\right]\sin\theta\,d\theta$$

$$-\frac{pa}{2t}\,2\pi a^2 t\int\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2\right]\sin\theta\,d\theta$$

The volume under the shell

$$= \frac{2\pi a^3}{3}\int \left[1+\frac{w}{a}\right]^3\left[\sin\theta\left[1-\frac{1}{2}\left(\frac{u}{a}\right)^2\right] + \cos\theta\left(\frac{u}{a}\right)\right]\left[1+\frac{1}{a}\frac{du}{d\theta}\right]d\theta$$

$$= \frac{2\pi a^3}{3}\int \left\{1+3\frac{w}{a}+3\left(\frac{w}{a}\right)^2\right\}\left\{\left[1-\frac{1}{2}\left(\frac{u}{a}\right)^2\right]\sin\theta + \left(\frac{u}{a}\right)\cos\theta\right\}\left\{1+\frac{1}{a}\frac{du}{d\theta}\right\}d\theta$$

$$\sim \frac{2\pi a^3}{3}\int\left[\sin\theta\left(1+3\frac{w}{a}+3\frac{w^2}{a^2}\right) - \frac{1}{2}\left(\frac{u}{a}\right)^2\sin\theta + \left(\frac{u}{a}\right)\cos\theta + 3\cos\theta\,\frac{uw}{a^2}\right]\left[1+\frac{1}{a}\frac{du}{d\theta}\right]d\theta.$$

$$\sim \frac{2\pi a^3}{3}\int\left[\sin\theta\left(3\frac{w}{a}+3\frac{w^2}{a^2}\right) - \frac{1}{2}\left(\frac{u}{a}\right)^2\sin\theta + \left(\frac{u}{a}\right)\cos\theta + 3\cos\theta\,\frac{uw}{a^2} + \frac{1}{a}\frac{du}{d\theta}\left(\sin\theta + 3\sin\theta\,\frac{w}{a} + \left(\frac{u}{a}\right)\cos\theta\right)\right]d\theta.$$

The integral to be minimized is

$$\left\{\frac{t}{a}\frac{E}{1-\nu^2}\right\}\left[\frac{1}{2}\int\left\{\left(\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right)^2+\left(\frac{w}{a}+\frac{u}{a}\cot\theta\right)^2+2\nu\left(\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right)\left(\frac{w}{a}+\frac{u}{a}\cot\theta\right)\right\}\right.$$

$$\sin\theta\, d\theta$$

$$\cancel{-\frac{p}{2}\int\left\{\frac{2w}{a}+\cos\theta\frac{u}{a}+\frac{1}{a}\frac{du}{d\theta}+\frac{1}{2}\left(\frac{u}{a}\right)^2+\frac{1}{a^2}\left(\frac{dw}{d\theta}\right)^2+\frac{uw}{a^2}\cos\theta\right\}\sin\theta\, d\theta}$$

$$-\frac{p}{3}\int\left\{\frac{3w}{a}+3\left(\frac{w}{a}\right)^2-\frac{1}{2}\left(\frac{u}{a}\right)^2+\cos\theta\left(\frac{u}{a}\right)+3\cos\theta\frac{uw}{a^2}\right.$$

$$\left.+\frac{1}{a}\frac{du}{d\theta}\left(1+3\frac{w}{a}+\cos\theta\frac{u}{a}\right)\right\}\sin\theta\, d\theta \qquad \pi_1$$

The integral to be minimized is

$$\left(\frac{t}{a}\frac{E}{1-\nu^2}\right)\left[\frac{1}{2}\int\left\{\left[\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right]^2+\left[\frac{w}{a}+\frac{u}{a}\cot\theta\right]^2+2\nu\left(\frac{1}{a}\frac{du}{d\theta}+\frac{w}{a}\right)\left(\frac{w}{a}+\frac{u}{a}\cot\theta\right)\right\}\sin\theta\, d\theta\right.$$

$$-\rho\int\left\{\frac{2w}{a}+\frac{5}{6}\cos\theta\frac{u}{a}+\frac{5}{6}\frac{1}{a}\frac{du}{d\theta}-\frac{5}{12}\left(\frac{u}{a}\right)^2+\frac{1}{2a^2}\left(\frac{dw}{d\theta}\right)^2\right.$$

$$\left.+\frac{3}{2}\frac{uw}{a^2}\cos\theta+\cos\theta\frac{1}{a^2}u\frac{du}{d\theta}\right\}\sin\theta\, d\theta.$$

$$-p\int\left\{\frac{w}{a}+\left(\frac{w}{a}\right)^2-\frac{1}{6}\left(\frac{u}{a}\right)^2+\frac{1}{3}\left(\frac{u}{a}\right)\cos\theta+\frac{uw}{a^2}\cos\theta+\frac{1}{a}\frac{du}{d\theta}\left(\frac{1}{3}+\frac{w}{a}+\cot\theta\frac{u}{3a}\right)\right\}$$

$$\sin\theta\, d\theta.$$

Bending without extension in meridian

We have the differential equations

$$\frac{dN_x'}{d\theta} + (N_x' - N_y')\cot\theta - Q_x - \frac{qa}{2}\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0$$

$$\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x' + N_y' + qa\left(\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta - \frac{2w}{a}\right)$$

$$- \frac{qa}{2}\left(\frac{du}{a\,d\theta} + \frac{d^2w}{a\,d\theta^2}\right) - \frac{qa}{2}\cot\theta\left(\frac{u}{a} + \frac{dw}{a\,d\theta}\right) = 0$$

$$\frac{dM_x}{d\theta} + (M_x - M_y)\cot\theta - Q_x a = 0.$$

Putting $w = \frac{du}{d\theta}$

$$N_x' = \frac{Eh}{1-\nu^2}\left\{\nu\left(\frac{u\cot\theta}{a} - \frac{du}{a\,d\theta}\right)\right\}$$

$$N_y' = \frac{Eh}{1-\nu^2}\left\{\frac{u\cot\theta}{a} - \frac{du}{a\,d\theta}\right\}$$

$$M_x' = -\frac{D}{a^2}\left[\frac{du}{d\theta} + \frac{d^3u}{d\theta^3} + \nu\left(u + \frac{d^2u}{d\theta^2}\right)\cot\theta\right]$$

$$M_y = -\frac{D}{a^2}\left[\left(u + \frac{d^2u}{d\theta^2}\right)\cot\theta + \nu\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\right]$$

23)

$$\frac{dN_x'}{d\theta} + (N_x' - N_y')\cot\theta - Q_x - \frac{qa}{2}\left(\frac{u}{a} + \frac{1}{a}\frac{d^2u}{d\theta^2}\right) = 0.$$

$$\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x' + N_y' + qa\Big[-\frac{du}{a\,d\theta} + \frac{u}{a}\cot\theta - \frac{1}{2}\frac{du}{a\,d\theta} - \frac{1}{2}\frac{d^3u}{a\,d\theta^3}$$

$$-\frac{\cot\theta}{2}\left(\frac{u}{a} + \frac{d^2u}{a\,d\theta^2}\right)\Big] = 0.$$

or

$$\boxed{\frac{dQ_x}{d\theta} + Q_x\cot\theta + N_x' + N_y' + \frac{qa}{2}\Big[\cot\theta\left(\frac{u}{a} - \frac{d^2u}{a\,d\theta^2}\right) - 3\frac{du}{a\,d\theta} - \frac{d^3u}{a\,d\theta^3}\Big] = 0.}$$

$$Q_x = \frac{1}{a}\left\{\frac{dM_x}{d\theta} + (M_x - M_y)\cot\theta\right\}$$

$1 + \cot^2\theta = \csc^2\theta$

$$\cancel{\frac{1}{a}\Big\{\frac{d^2M_x}{d\theta^2} + \Big(\frac{dM_x}{d\theta} - \frac{dM_y}{}}$$

$$= -\frac{D}{a^3}\Big\{\frac{d^2u}{d\theta^2} + \frac{d^4u}{d\theta^4} + \nu\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\cot\theta - \nu\left(u + \frac{d^2u}{d\theta^2}\right)\csc^2\theta$$

$$- \Big\{(1-\nu)\cot\theta\left(u - \frac{du}{d\theta} + \frac{d^2u}{d\theta^2} - \frac{d^3u}{d\theta^3}\right)\Big\}$$

$$Q_x = -\frac{D}{a^3}\Big\{\frac{d^2u}{d\theta^2} + \frac{d^4u}{d\theta^4} + \cancel{\nu\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\cot\theta} - \nu\left(u + \frac{d^2u}{d\theta^2}\right)\csc^2\theta$$

$$-\cot\theta\left(u - \frac{du}{d\theta} + \frac{d^2u}{d\theta^2} - \frac{d^3u}{d\theta^3}\right) + \nu\left(u - \cancel{\frac{du}{d\theta}} + \frac{d^2u}{d\theta^2} - \cancel{\frac{d^3u}{d\theta^3}}\right)\cot\theta\Big\}$$

$$= -\frac{D}{a^3}\Big\{\frac{d^2u}{d\theta^2} + \frac{d^4u}{d\theta^4} - \nu\left(u + \frac{d^2u}{d\theta^2}\right)\csc^2\theta - (1-\nu)\cot\theta\left(u + \frac{d^2u}{d\theta^2}\right) + \cot\theta\left(\frac{du}{d\theta} + \frac{d^3u}{d\theta^3}\right)\Big\}$$

24)

$$-\frac{D}{a^3}\Big[\frac{d^3u}{d\theta^3}+\frac{d^5u}{d\theta^5}-\nu\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)\csc^2\theta+\nu\Big(u+\frac{d^2u}{d\theta^2}\Big)\csc^2\theta\cdot\cot\theta$$

$$+(1-\nu)\quad\Big(u+\frac{d^2u}{d\theta^2}\Big)-(1-\nu)\cot\theta\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)$$

$$-\quad\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)+2\cot\theta\Big(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\Big)$$

$$+\cancel{\cot\theta\Big(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\Big)-\nu\Big(u+\frac{d^2u}{d\theta^2}\Big)\csc^2\theta\cot\theta}$$

$$-\cancel{(1-\nu)\cot^2\theta\Big(u+\frac{d^2u}{d\theta^2}\Big)}+\cot^2\theta\cancel{\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)}\Big]$$

$$+\frac{Eh}{1-\nu^2}(1+\nu)\Big[\frac{u\cot\theta}{a}-\frac{du}{a\,d\theta}\Big]+\frac{qa}{2}\Big[\cot\theta\Big(\frac{u}{a}-\frac{d^2u}{a\,d\theta^2}\Big)$$

$$-3\frac{du}{a\,d\theta}-\frac{d^3u}{a\,d\theta^3}\Big]=0$$

$$-\frac{D}{a^3}\Big[\frac{d^3u}{d\theta^3}+\frac{d^5u}{d\theta^5}-(1+\nu)\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)-\Big\{(1-\nu)\cot\theta+\nu\cot^2\theta\Big\}\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)$$

$$+2\cot\theta\Big(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\Big)+\Big\{(1-\nu)+\nu(\cot\theta+\cot^3\theta)\Big\}\Big(u+\frac{d^2u}{d\theta^2}\Big)\Big]$$

$$+\frac{Eh}{1-\nu^2}(1+\nu)\Big[\frac{u\cot\theta}{a}-\frac{du}{a\,d\theta}\Big]+\frac{qa}{2}\Big[\cot\theta\Big(\frac{u}{a}-\frac{d^2u}{a\,d\theta^2}\Big)$$

$$-3\frac{du}{a\,d\theta}-\frac{d^3u}{a\,d\theta^3}\Big]=0$$

$$-\alpha\Big[\frac{d^3u}{d\theta^3}+\frac{d^2u}{d\theta^2}-(1+\nu)\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)-\{(1-\nu)\cot\theta+\nu\cot^3\theta\}\Big(\frac{du}{d\theta}+\frac{d^3u}{d\theta^3}\Big)$$

$$+2\,[illegible]\Big(\frac{d^2u}{d\theta^2}+\frac{d^4u}{d\theta^4}\Big)+\{(1-\nu)+\nu(\cot\theta+\cot^3\theta)\}\Big(u+\frac{d^2u}{d\theta^2}\Big)\Big]$$

$$+(1+\nu)\Big[u\cot\theta-\frac{du}{d\theta}\Big]+\phi\Big[\Big(u-\frac{d^2u}{d\theta^2}\Big)\cot\theta-3\frac{du}{d\theta}-\frac{d^3u}{d\theta^3}\Big]=0.$$

Put $u=\dfrac{d\psi}{d\theta}$

$$-\alpha\Big[\frac{d^6\psi}{d\theta^6}+2\cot\theta\frac{d^5\psi}{d\theta^5}-\{\nu\csc^2\theta+(1-\nu)\cot\theta\}\frac{d^4\psi}{d\theta^4}$$

$$+\{(1-\nu)+(2+\nu)\cot\theta+\nu\cot^3\theta\}\frac{d^3\psi}{d\theta^3}$$

$$-\{(1+\nu)+(1-\nu)\cot\theta+\nu\cot^2\theta\}\frac{d^2\psi}{d\theta^2}$$

$$+\{(1-\nu)+\nu(\cot\theta+\cot^3\theta)\}\frac{d\psi}{d\theta}\Big]$$

$$+(1+\nu)\Big[\cot\theta\frac{d\psi}{d\theta}-\frac{d^2\psi}{d\theta^2}\Big]+\phi\Big[\Big(\frac{d\psi}{d\theta}-\frac{d^3\psi}{d\theta^3}\Big)\cot\theta-3\frac{d^2\psi}{d\theta^2}-\frac{d^4\psi}{d\theta^4}\Big]=0$$

<u>Pure Bending</u> Change in curvature $\frac{1}{a}$

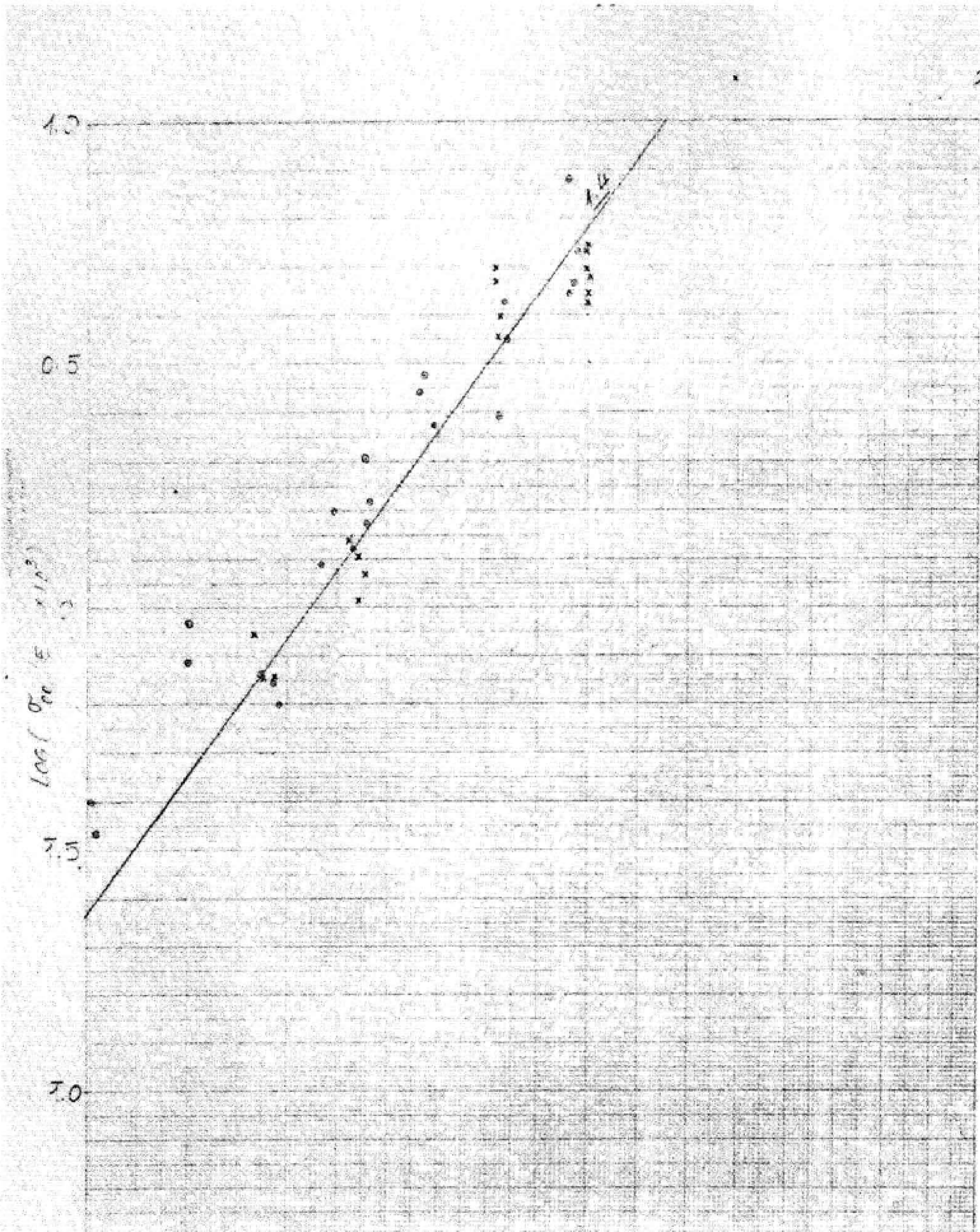
Strain energy $\sim$ (Change in curvature)^2 bending stiffness $\times$ area $\sim$ $\left(\frac{1}{a}\right)^2 Et^3\cdot a^2 \sim Et^3$

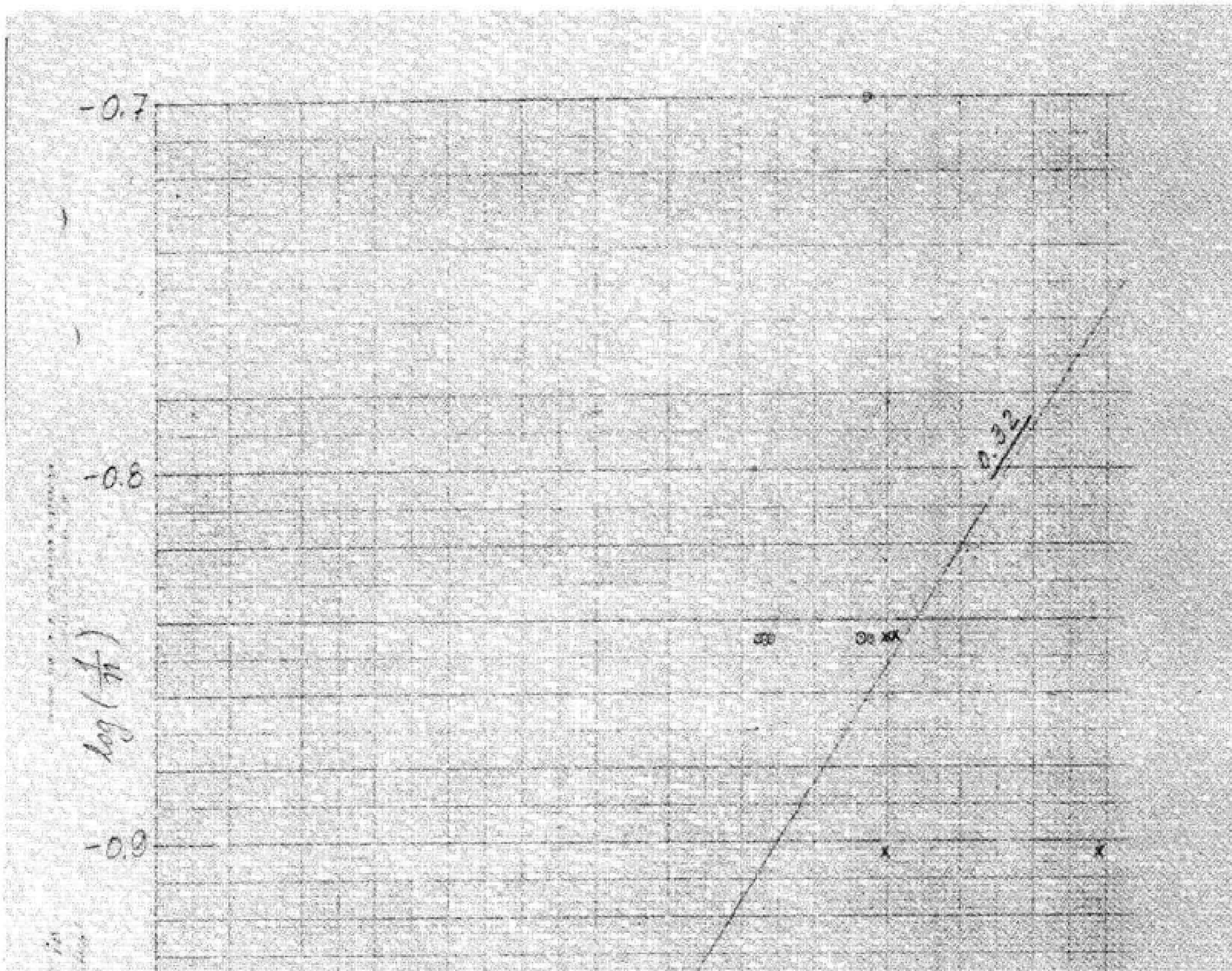
Potential energy $\sim p\,a^3$

$p_{cr} \sim E\left(\frac{t}{a}\right)^3$ $\qquad$ $\sigma_{cr} \sim E\left(\frac{t}{a}\right)^2$

	Steel			Brass			
	$\frac{t}{a}$	A	σ_{cr}/E	$\frac{t}{a}$	A	σ_{cr}/E	
1	1.015×10^{-3}	0.00512	1.484×10^{-4}	2.060×10^{-3}	0.01037	3.96×10^{-4}	27
2	0.981×10^{-3}	0.00494	1.586×10^{-4}	3.180×10^{-3}	0.00701	4.40×10^{-3}	28
3	1.556×10^{-3}	0.00334	2.355×10^{-4}	3.15×10^{-3}	0.00694	4.21×10^{-3}	29
4	1.483×10^{-3}	0.00327	2.99×10^{-4}	6.19×10^{-3}	0.00345	12.12×10^{-3}	30
5	3.012×10^{-3}	0.00168	5.36×10^{-4}	1.05×10^{-3}	0.00530	1.38×10^{-3}	31
6	2.885×10^{-3}	0.00161	7.58×10^{-4}	3.13×10^{-3}	0.00175	4.94×10^{-3}	32
7	0.765×10^{-3}	0.00386	0.63×10^{-4}	3.16×10^{-3}	0.00175	5.54×10^{-3}	33
8	0.744×10^{-3}	0.00375	0.70×10^{-4}	0.744×10^{-3}	0.00376	0.72×10^{-3}	34
9	1.162×10^{-3}	0.00257	1.65×10^{-4}	0.710×10^{-3}	0.00357	0.72×10^{-3}	35
10	1.137×10^{-3}	0.00250	2.03×10^{-4}	1.136×10^{-3}	0.00251	1.17×10^{-3}	36
11	2.175×10^{-3}	0.00121	3.54×10^{-4}	1.10×10^{-3}	0.00244	1.03×10^{-3}	37
12	2.155×10^{-3}	0.00120	4.23×10^{-4}	2.09×10^{-3}	0.01060	3.60×10^{-3}	38
13	0.932×10^{-3}	0.00470	1.23×10^{-4}	2.06×10^{-3}	0.01044	5.02×10^{-3}	39
14	1.460×10^{-3}	0.00322	2.75×10^{-4}	2.06×10^{-3}	0.01044	4.79×10^{-3}	40
15	2.906×10^{-3}	0.00162	4.46×10^{-4}	3.19×10^{-3}	0.00704	4.77×10^{-3}	41
16	0.702×10^{-3}	0.00354	0.73×10^{-4}	6.23×10^{-3}	0.00347	14.18×10^{-3}	42
17	1.071×10^{-3}	0.00237	1.32×10^{-4}	0.68×10^{-3}	0.00343	0.88×10^{-3}	43
18	2.970×10^{-3}	0.00166	4.60×10^{-4}	1.10×10^{-3}	0.00244	1.28×10^{-3}	44
19	2.110×10^{-3}	0.00118	2.46×10^{-4}	2.09×10^{-3}	0.01052	3.56×10^{-3}	45
20	0.5×10^{-3}		0.15×10^{-4}	3.15×10^{-3}	0.00696	5.40×10^{-3}	46
21	"		0.77×10^{-4}	6.34×10^{-3}	0.00353	14.74×10^{-3}	47
22	0.33×10^{-3}		0.342×10^{-4}				48
23	0.322×10^{-3}		0.400×10^{-4}				49
24							50
25							51
26							52

28)




-0.7
-0.8
-0.9
-1.0
-1.1
$\log\left(\frac{f}{n}\right)$
0.32
0
0.5

1)

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